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Weakly renormalized near 1/16 SUSY Fermi liquid operators in $\mathcal{N} = 4$ SYM

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Abstract: We discuss a class of Fermi Liquid Operators in $\mathcal{N} = 4$ SYM. We show that these operators are eigenstates of the full quantum dilatation operator. We compute their 1 and 2 loop anomalous dimensions, and show that, similar to Fermi liquids in condensed matter systems, these corrections are suppressed by an arbitrarily small parameter, which is the equivalent of one over the Fermi energy. These operators are, at the classical level, descendants of 1/16 BPS operators, with some scaling properties similar to those of the 1/16 Black Holes in AdS_5.

Keywords: Black Holes in String Theory, AdS-CFT Correspondence, Supersymmetry and Duality.

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1. Introduction

In recent years we have witnessed considerable progress in understanding $\mathcal{N}=4$ SYM away from the safe shores of the (chiral) BPS spectrum. An important trigger to this progress was the development of tools for perturbative calculations on both sides of the AdS/CFT duality [1−3]. On the CFT side spin chains [4, 5], integrability [6, 7] and unitarity techniques [8] made exact results, to all orders in perturbation theory, possible, such as the BES and FRS equations [9, 10] and BDS $n=4,5$ amplitudes [11]. In the AdS side the BMN limit [12, 13], semiclassical quantization of rotating superstrings [14], integrability of the string worldsheet [15, 16] and semiclassical solution of strings stretching to the boundary [17] permit the quantitative understanding of similar results at strong coupling. In this paper we suggest that Fermi liquid like operators might be a new avenue where weak and strong coupling results can be compared (and in particular, in the context of low SUSY black holes).

A key characteristic of the recent tools mentioned above is a proximity to unitarity bounds which reduces the complexity of the calculation. For example the $\mathfrak{su}(2)$ spin chain captures the excitations of the theory near a 1/2-BPS unitarity bound

$$\Delta_1 := d - \frac{1}{2}q_1 - p - \frac{3}{2}q_2 \geq 0 \ , \qquad \Delta_2 := d - \frac{3}{2}q_1 - p - \frac{1}{2}q_2 \geq 0\,,$$

where d is the dimension of the state, (q_1, p, q_2) are SU(4) charges (using Dynkin labels) and the angular moneta is set to zero. The ground state of the spin chain has $q_1 = q_2 = 0$, $d = p$ and saturates both bounds. The possible excitation of this spin chain at weak coupling (below an integer gap in dimension) are made out of a complex scalar with weight $(q_1, p, q_2) = (1, -1, 1)$. In perturbation theory states in the $\mathfrak{su}(2)$ spin chain can only mix within themselves since the quantities Δ_1, Δ_2 are continuous in $g_{\rm ym}$. The reduced complexity is present also in the AdS side where the corresponding states come from quantization near a string with angular momentum in the S^5 (for p scaling as $\sqrt{N}$ this is the BMN limit).

The spin chain techniques, and related ones, have proven to be of great use in the BPS sector or close to it. As one reduces the amount of supersymmetry, the level of complexity increases up to the full non-SUSY case, where few precise quantitative matches exist. In particular, the natural intermediate stage of the 1/16-BPS spectrum, and the sector of states near the corresponding unitarity bounds, is still out of our reach. For example, we possess some knowledge of the sector in the free $\mathcal{N} = 4$ theory [18] and in the classical gravity [19] limits, however the interpolation between these limits has seen little progress (see [20 – 23] for recent discussions).

1/16 black holes. The problem of the 1/16 operators is particularly interesting since in the bulk these are identified as black holes [19]. This in contrast to operators with higher SUSY, of which they are not enough to form a black hole. This is an indication that these black holes, although SUSY, are sensitive to the full N^2 degrees of freedom of $\mathcal{N} = 4$ SYM and can be a useful intermediate stage to the understanding of the more generic states of $\mathcal{N} = 4$ SYM. Furthermore, one can argue that similar black holes in AdS_4 are sensitive to the (in)famous $N^{3/2}$ degrees of freedom in this field.

Fermi-sea model of the 1/16 black holes. In [20] we pointed out that Fermi-sea operators in $\mathcal{N} = 4$ SYM may play a key role in the construction of 1/16-BPS states in the interacting theory. The smoking gun is a scaling between charge and angular momentum found in the black-holes solutions (where $(q_1, p, q_2) = (0, 0, q)$)

$$\begin{aligned} j_R/N^2 &\propto \left(q/N^2\right)^2 && \text{if} \quad j_L = j_R && q \gg N^2 \\ j_R/N^2 &\propto \left(q/N^2\right)^{3/2} && \text{if} \quad j_L = 0 && q \gg N^2. \end{aligned} \tag{1.1}$$

These relation are typical to Fermi-sea models where each fermion carries charge $q = 1$ and the levels are graded by the angular momentum j_R. Filling the levels up to some Fermi-level K we find a relation

$$(j_R/N^2) \propto (q/N^2)^{(m+1)/m}\,, \tag{1.2}$$

with m is the dimensionality of the Fermi-sea for, and N^2 is the degeneracy of fermions. $j_L = j_R$ is naturally associated with $m = 1$ and $j_L = 0$ is naturally associated with $m = 2$.

Even though the scaling is similar, the precise coefficients in front of the relations are different for the Fermi sea operators and the Black holes. We are therefore still missing some components in the operators that correspond to the black holes, but we can still ask what are the algebraic properties of the Fermi sea by themselves, as these properties may carry over to the full operators that correspond to the black holes.

In particular, the operators here are not primaries, and they are not precisely the same operators discussed in [20], where an attempt to find BPS primaries was carried out. However, they are very similar to the latter in the sense that we can go to the operators of [20] by adding an additional insertion of the gauge field strength (and additional insertions of scalars to go to the general case discussed there). Without this insertion, which is the case here, the operators cannot be BPS primaries as they does not satisfy the unitarity bound ($d = 2 + 2j_L + \frac{1}{2}q_1 + p + \frac{3}{2}3q_2$), and indeeds the operators are descendants of non-BPS operators.

We would also like to emphasize that the results of [20] were limited to tree-level (with weak coupling). The tools discussed in this work can be used to refine the calculations there to include 1-loop contributions. In [20] we argued that certain operators cannot be written as Q of something, but we did not identify which combination was primary. In order to do so, and verify or refute the construction there using the tools in the current paper, one still needs to solve an operator mixing problem. More generally, one could ask a similar question on the GR side. I.e., whether the 1/16 SUSY BH are exactly SUSY, or whether SUSY is lifted by some higher order correction, which translates to whether there is really a large degeneracy of SUSY operators or not. Currently we believe that both option are viable, but checking which is true requires a complicated calculation. However we choose to postpone this investigation to future work and rather take a new direction.

Fermi-sea and Fermi liquids. The Fermi-sea operators/states are created by fermions belonging to the $\mathfrak{su}(1,1)$ sector of $\mathcal{N} = 4$ SYM. In principle they are similar to Fermi-sea states common in condensed matter physics, however our fermions come in an $N^2 - 1$-fold degeneracy, arising from the SU(N) gauge group. Another peculiarity of our Fermi-sea is that it carries large angular momentum, and therefore states are not graded and filled only by energy, which leads to a filling which is the isotropic, but rather the momentum shells which are filled are confined to 1-dim "lightcone" in momentum space (in the $j_L = j_R$ case, which will be our main focus). Despite the strange nature of the fermions we will demonstrate that they form a normal Fermi liquid just the same way interacting fermions do in earthly physics.

In a sense, the appearance of Fermi-sea operators in the AdS/CFT correspondence could be expected, since they are the ground state of choice of many condensed matter system. Let us review briefly the notion of normal Fermi liquid. The phenomenological theory of Landau [24][1] describes the state of strongly coupled fermions at low temperature. This theory, originally constructed for liquid ^{3}He, was found to be a good description

[1]See [25] for a comprehensive review.

(generalized for charged fermions) in much more general situations ranging from electron in metals (when not undergoing a superconductor phase transition, such as Alkali metals) to highly dense matter in stars.

Landau [24] considered a non-interacting gas of fermions at equilibrium. At high temperatures T the distribution of particles with energy ϵ is given by the well known Fermi-Dirac distribution. In this regime, the kinetic term usually dominant, and any interaction between the fermions is treated as a perturbation. As the temperature is lowered, $T \to 0$, the Fermi-Dirac distribution approaches a step function at the Fermi-energy (fixed by the chemical potential). At the same time a critical temperature appears when interactions between the fermions become important. Fermion systems where the particle interaction and the exclusion principle act simultaneously are often named degenerate Fermi liquids.

The degenerate Fermi liquid at $T = 0$ is described by a Fermi-sea; it's elementary excitations are quasiparticles and quasiholes in analogy to the non-interacting case. However the nature of the quasiparticles could be very different from the excitation of the non-interacting system. Landau's phenomenological model describes the energy of a the degenerate Fermi liquid in terms of an the quasi-particle density, the effective Fermi velocity (effective mass) and effective couplings (known as Landau's parameters)

$$E = \sum_k v_F^*(\mid k \mid - k_F)\delta n(k) + \frac{1}{2}\sum_{k,k'} f^*(\mid k - k' \mid)\delta n(k)\delta n(k') \,, \tag{1.3}$$

with $\delta n(k)$, is the density of quasi-particles, exhibiting a Fermi-Dirac distribution in equilibrium (with zero chemical potential). In principle, the various couplings can be computed by flowing from the UV, but, as expected, this is usually difficult in practice. However, it is often the case that the excitation of quasi-particles near the Fermi-surface are long-lived[2] allowing to treat them as stable excitation and allowing for a perturbative expansion in T/E_f. It is quite remarkable that this simple and intuitive model, and the introduction of the T/E_f perturbation theory, are strong enough to describe all the observed phenomena of ^{3}He, and other normal Fermi liquids.

The main obstruction for the appearance of a normal Fermi liquid is a breaking of symmetry. The most famous examples are the BCS mechanism (where a phonon-electron interaction drives a condensation of Copper pairs) and the Wigner crystal (where the translation symmetry breaks at low densities).

In this work we analyze the fate of the Fermi liquid operators of $\mathcal{N} = 4$ SYM in perturbation theory. We show that

- Based on symmetries, we prove that the Fermi-sea state is an eigenvalue of the dilatation operators to all order in perturbation theory and does not mix with any other operator in $\mathcal{N} = 4$ SYM.
- We compute its 1 and 2 loop anomalous dimension and find a non-trivial cancelation at weak coupling leading to a much milder anomalous dimension compared to the

[2] Due to the interaction any quasiparticles, above the Fermi surface, can decay into two quasiparticles closer to the Fermi surface, however the decay rate behaves as $1/(E_{q.p} - E_F)^2$.

naive expectation from familiar results of spin chain states. The corrections to the classical dimension are suppressed by one over the analogue of the Fermi energy, suggesting that the operators can be traced from weak to strong coupling.

- Investigating quasi-particles energies near the Fermi-sea, we find that the cancelation allows us to write a consistent Landau Fermi liquid model (1.3) with a well defined $1/E_f$ expansion.

The $1/E_f$ expansion sheds new light on the problem of 1/16 BPS operators. One route is to continue the search for a large degeneracy of operators, which remain 1/16 BPS when field theoretic loop correction are taken into account. However a new scenario is possible which suggests that SUSY and the degeneracy are lifted by α' or stringy loop corrections.

The paper is organized as follows. In section 2 we recap some notations and tools for $\mathcal{N}=4$ SYM . In section 3 we discuss the fermionic $\mathfrak{su}(1,1)$ sector of $\mathcal{N}=4$ and the 1-dim Fermi-sea operators within weak coupling perturbation theory. In section 4 we show that indeed the Fermi-sea operator is a ground state of a Fermi liquid. In section 5 we briefly discuss a possible generalization of the work to a 2-dim Fermi-sea. We conclude in section 6 with some additional discussion and directions for future research.

2. Tools for $\mathcal{N}=4$ super Yang-Mills theory

In this section we briefly recall some features of $\mathcal{N}=4$ SYM and it's $\mathfrak{psu}(2,2|4)$ algebra, which we will need later on. We follow the notation of [7] - readers familiar with this work may skip this section.

The component of the $\mathcal{N}=4$ multiplet are

- The gauge field strength $F_{\alpha\beta}$ and $\bar{F}_{\dot{\alpha}\dot{\beta}}$.
- The gauginos $\Psi_{\alpha a}$ and $\bar{\Psi}^a_{\dot{\alpha}}$.
- The complex scalars Φ_{ab} with the antisymmetry $\Phi_{ab}=-\Phi_{ba}$.

The undotted Greek letters $(\alpha,\beta,\ldots)$, dotted Greek letters $(\dot{\alpha},\dot{\beta},\ldots)$ and Latin letters $(a,b\ldots)$ stands for $\mathrm{SU}(2)_L$, $\mathrm{SU}(2)_R$ and SU(4) fundamental indices. Raising and lowering the SU(4) indices changes between the fundamental and anti-fundamental representations. The scalars belong to a **6** of SU(4) and obey the reality condition

$$(\Phi_{ab})^\dagger := \bar{\Phi}^{ab} = \frac{1}{2}\epsilon^{abcd}\Phi_{cd}\ . \tag{2.1}$$

The gauge group is $G=\mathrm{SU}(N)$, and all fields transform in the adjoint representation. When we will need to be specific about the gauge group structure we will write all fields as $\mathcal{W}=\mathcal{W}^a t^a$ with $a=1,\ldots\dim G$, and t^a are generators of $\mathrm{SU}(N)$.[3] The covariant derivative is

$$D_{\alpha\dot{\alpha}}\mathcal{W} := (\sigma^\mu)_{\alpha\dot{\alpha}}\left(\partial_\mu\mathcal{W}-i[A_\mu,\mathcal{W}]\right)\ . \tag{2.2}$$

[3]We use the same letters for gauge group and SU(4) indices, it will be clear to distinguish between them from the context.

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We use the fundamental representation for SU(N) with the following properties

$$\operatorname{Tr} t^a t^b = \delta^{ab} , \qquad \sum_a t^a t^a = \frac{N^2-1}{N}\mathbb{I} . \tag{2.3}$$

We use the coupling constat g^2 proportional to the 't Hooft coupling

$$g^2 = \frac{g_{\rm ym}^2 N}{8\pi^2} . \tag{2.4}$$

The oscillator representation and spin chains. In the planar limit a useful set of states are spin chains. These are states which are mapped to single trace local operators

$$| \mathcal{W}_1\mathcal{W}_2 \ldots \mathcal{W}_n \rangle \mapsto \operatorname{Tr}(\mathcal{W}_1(x)\mathcal{W}_2(x)\ldots\mathcal{W}_n(x)) . \tag{2.5}$$

For spin chains all the gauge group information is encoded in the location inside the chain (relative to other spins). This allows one to write each spin using the oscillator representation. There are three types of oscillators - left-handed bosons, right-handed bosons and SU(4) charged fermions - which satisfy:

$$\left[\mathbf{a}^\alpha, \mathbf{a}^\dagger_\beta\right] = \delta^\alpha_\beta , \qquad \left[\mathbf{b}^{\dot\alpha}, \mathbf{b}^\dagger_{\dot\beta}\right] = \delta^{\dot\alpha}_{\dot\beta} , \qquad \left\{\mathbf{c}^a, \mathbf{c}^\dagger_b\right\} = \delta^a_b . \tag{2.6}$$

Using these oscillators one can build all possible spins

$$\begin{aligned}
D^k F &:= (\mathbf{a}^\dagger)^{k+2} (\mathbf{b}^\dagger)^k \quad (\mathbf{c}^\dagger)^0 \mid 0 \rangle \\
D^k \Psi &:= (\mathbf{a}^\dagger)^{k+1} (\mathbf{b}^\dagger)^k \quad (\mathbf{c}^\dagger)^1 \mid 0 \rangle \\
D^k \Phi &:= (\mathbf{a}^\dagger)^{k} \quad (\mathbf{b}^\dagger)^k \quad (\mathbf{c}^\dagger)^2 \mid 0 \rangle \\
D^k \bar\Psi &:= (\mathbf{a}^\dagger)^{k} \quad (\mathbf{b}^\dagger)^{k+1} (\mathbf{c}^\dagger)^3 \mid 0 \rangle \\
D^k \bar F &:= (\mathbf{a}^\dagger)^{k} \quad (\mathbf{b}^\dagger)^{k+2} (\mathbf{c}^\dagger)^4 \mid 0 \rangle .
\end{aligned} \tag{2.7}$$

The $\mathfrak{psu}(2,2|4)$ algebra of the zero coupling theory is made out of bilinears of oscillators (see appendix A for details). The generator of $\mathfrak{psu}(2,2|4)$ are expected to receive quantum correction, it is possible to choose a regularization scheme such that operator mixing occurs only between operators with the same zero-coupling dilatation (dimension). Using these scheme all operators can be written as a formal expansion in g

$$\mathcal{X}(g) = \sum_{n=0}^{\infty} \mathcal{X}_n g^n , \qquad [\mathfrak{D}_0, \mathcal{X}_n] = d_{\mathcal{X}} \mathcal{X}_n , \qquad \mathcal{X} \in \mathfrak{psu}(2,2|4) . \tag{2.8}$$

All these operators depend of course on N - the rank of the gauge group - but we will suppress this dependence in the notation.

The anomalous dimension operator $\delta\mathfrak{D} := \mathfrak{D} - \mathfrak{D}_0$ commutes with all the $\mathfrak{psu}(2,2|4)$ algebra. It has an expansion

$$\delta\mathfrak{D} = \sum_{n=2}^{\infty} \delta\mathfrak{D}_n g^n . \tag{2.9}$$

For spin-chains in the planar limit the leading term of the anomalous dimension is given by the 'Harmonic action'

$$\delta\mathfrak{D}_2 = \sum_{k=0}^{L} H_{k,k+1}\,, \qquad H_{1,2} \,|\, AB\,\rangle = \sum_{C,D} c_{n,n_{12},n_{21}} \,|\, CD\,\rangle\,, \tag{2.10}$$

where L the length of the spin chain, and k runs over the spin location in the chain. The two-spin operator $H_{1,2}$ is evaluated by considering all possible two spin states generated by having an oscillator hop from one site to the other. The coefficient $c_{n,n_{12},n_{21}}$ is a function of the total number of oscillator in the two spins (conserved by $H_{1,2}$), n_{12} the number of spins jumping form site 1 to 2 and n_{21} the number of spins jumping from site 2 to 1

$$c_{n,n_{12},n_{21}} = \begin{cases} (-)^{1+n_{12}n_{21}} \frac{\Gamma\left(\frac{1}{2}(n_{12}+n_{21})\right)\Gamma\left(1+\frac{1}{2}(n-n_{12}-n_{21})\right)}{\Gamma\left(1+\frac{1}{2}n\right)} & n_{12}+n_{21}>0 \\ h(n) & n_{12}=n_{21}=0 \end{cases} \tag{2.11}$$

where $h(n)$ are the harmonic numbers

$$h(n) := \sum_{k=1}^{n} \frac{1}{k}\,, \qquad h(0) := 0\ .$$

The anomalous dimension operator and closed sectors. The planar expression can be lifted to the full theory at finite N. Using Feynman diagrams it can be proven that the leading term (in g^2) in $\delta\mathfrak{D}$ has the general form

$$\delta\mathfrak{D}_2 = -\frac{1}{N} \sum_{A,B,C,D} \mathcal{C}^{AB}_{CD}\,\mathrm{Tr} :\left[\mathcal{W}_A, \check{\mathcal{W}}^C\right]\left[\mathcal{W}_B, \check{\mathcal{W}}^D\right]:\,, \tag{2.12}$$

where the summation is over all partons in the theory.[4] The symbol $\mathcal{W}_A$ stands for the parton and $\check{\mathcal{W}}^A$ are functional derivatives with respect to the partons

$$(\check{\mathcal{W}}^A)^a = \frac{\delta}{\delta(\mathcal{W}_A)^a}\,, \qquad a = 1,2\ldots \dim G\ . \tag{2.13}$$

The coefficient $\mathcal{C}^{AB}_{CD}$ are calculated directly from the Feynman diagrams. Computing $\delta\mathfrak{D}_2$ on the spin chain determines it completely. Indeed, in the full theory

$$\mathcal{C}^{AB}_{CD} = \frac{1}{2}(-)^{\zeta_C(\zeta_B+\zeta_C)}\Big\langle\, AB \,\Big|\, H_{12} \,\Big|\, CD\,\Big\rangle\,, \tag{2.14}$$

where the l.h.s. is the full expression and the r.h.s. is the planar expression (ζ_X is the fermion number, $\zeta_X = 1(0)$ for fermion(boson)). This is just the statement that at order g^2, all Feynman diagrams contributing to the anomalous dimension are planar. However for higher correction there are non-planar diagrams which are not captured by the spin chains.

In general the anomalous dimension operator mixes different operators (states). However one can use a regularization scheme where the Poincaré group and R-symmetry do not

[4] i.e., elementary fields acted by covariant derivatives, $\mathcal{W} = \left\{D^kF,\ D^k\Psi_i,\ D^k\Phi_{ij},\ D^k\bar{\Psi}^i,\ D^k\bar{F}\right\}$, with all lorentz indices symmetrized.

receive quantum corrections. In this scheme operator which have a definite charge under $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L \times \mathrm{SU}(4) \times \mathrm{U}(1)_{\mathfrak{D}_0}$ can mix only with operators with the same charges. A sector of the theory is a set of states closed under mixing, i.e cannot mix with states outside the sector, due to conservation of the above charges. An example is the $\mathfrak{sl}(2)$ sector which is generated by a single scalar and lightcone derivatives:

$$\mathcal{W} \in \left\{\Phi_{34},\ D_{1\dot{1}}\Phi_{34},\ (D_{1\dot{1}})^2\Phi_{34},\ \ldots\right\} . \tag{2.15}$$

All states in the sector has charges $(d_0; j_L, j_R; q_1, p, q_2; L) = (s+p; s, s; 0, p, 0; p)$. To prove that there are no other states in the theory with these charges one uses the fact that all parton of the theory obey $d_0 \geq j_L + j_R$. Among the partons with $d_0 = j_L + j_R$ only $(D_{1\dot{1}})^n\Phi_{34}$ has $q_1 + q_2 = 0$. The full set of closed sector and many more details may be found in [7, 26].

3. The Fermi liquid ground state

The Hilbert space of zero coupling $\mathcal{N} = 4$ SYM on S^3 is the Fock space generated by the mode expansion of the elementary fields on S^3, with the additional constraint of gauge invariance. Applying the operator-state mapping, these states correspond to local gauge invariant operators generated by the partons $\partial^k F, \partial^k\Psi_i, \partial^k\Phi_{ij}, \partial^k\bar{\Psi}^i$, and $\partial^k\bar{F}$. The Hamiltonian of the theory on S^3 is mapped to the dilatation operator, which in the notation discussed before and in $g = 0$ is

$$\mathcal{H} = \mathfrak{D} = \sum_A d_0 \,\mathrm{Tr}\, :\!\mathcal{W}_A \check{\mathcal{W}}^A\!: , \tag{3.1}$$

where d_0 is the classical dimension of $\mathcal{W}_A$. We often switch back and forth between energy of a state and the dilatation charge of a local operator, not to be confused with the energy in Minkowski space.

In a grand canonical ensemble the distribution functions is governed by the Gibbs free energy, which is just the Hamiltonian shifted by chemical potentials for a set of U(1) charges. The charges include all global internal symmetries as well as the angular momenta on the S^3

$$\beta\mathcal{F} = \beta\mathcal{H} - \sum_{I=1}^{5} \mu_I Q_I , \tag{3.2}$$

where β is the inverse temperature, the μ_i are the chemical potentials and Q_I are the corresponding charge operators. The ground state is the state (or a set of states) minimizing the free energy at zero temperature. We will consider twisted sector of the theory where $\mu_I \propto \beta$ (cf. [27]).

The Fermi-sea operators appear in the ensemble for a specific choice of chemical potentials, such that $\mathcal{F}$, localizes near a $\frac{1}{16}$-BPS bound, i.e

$$\mathcal{F} = \left(\mathfrak{D} - 2\mathfrak{J}_R^3 - \frac{1}{2}\mathfrak{q}_1 - \mathfrak{p} - \frac{3}{2}\mathfrak{q}_2\right) + O(\beta^{-1}) , \tag{3.3}$$

where $\mathfrak{J}_R^3$ is the right-handed angular momentum charge and $(\mathfrak{q}_1, \mathfrak{p}, \mathfrak{q}_2)$ are SU(4) charges (see appendix- A for the definition of these charges). For the specific choice above, all partons operators have a non-negative contribution to the free energy. The ground state in this sector is generated by partons which has vanishing contribution

$$\partial_{\alpha_1 \dot{1}} \partial_{\alpha_2 \dot{1}} \cdots \partial_{\alpha_n \dot{1}} \mathcal{X}, \qquad \mathcal{X} \in \left\{ \Psi_{\alpha_{n+1} 4}\,,\ \bar{F}_{\dot{1}\dot{1}}\,,\ \Phi_{i4}\,,\ \bar{\Psi}^i_{\dot{1}} \mid i = 1,2,3 \right\} . \tag{3.4}$$

Turning on weak Yang-Mills coupling generates anomalous dimension and mixing between the local operators. However, since the zero coupling dilatation charge is conserved by Feynman diagrams, the difference $d_0 - 2j_R^3 - \frac{1}{2}q_1 - p - \frac{3}{2}q_2$ is also conserved and operators generated from (3.4)[5] can mix only within themselves (due to the integer gap when inserting operators outside this sector). As a consequence the Hilbert space generated by the partons of (3.4) forms a closed sector of $\mathcal{N} = 4$ SYM. The partons of this sector form a representation of $\mathfrak{su}(1, 2|3)$ algebra, which is common to use as a name for the sector [7].

The anomalous dimensions, or mixing, do not vanish for all operator in the sector. Rather it is encoded in

$$\mathcal{F}_{(\mathfrak{su}(1,2|3))} = \left(\mathfrak{D} - 2\mathfrak{J}_R^3 - \frac{1}{2}\mathfrak{q}_1 - \mathfrak{p} - \frac{3}{2}\mathfrak{q}_2 \right) = (\mathfrak{D} - \mathfrak{D}_0) = \delta\mathfrak{D} \ .$$

An operator is a 1/16 BPS, at finite values of $g_{\rm ym}$, iff anomalous dimension vanishes ($\delta\mathfrak{D} = 0$). Such ground states of the $\mathfrak{su}(1, 2|3)$ sector are part of semi-short multiplets (see [23] for further discussion). Classifying all exact 1/16-BPS operators is a very interesting open problem.

In the following we will treat the anomalous dimension operator $\delta\mathfrak{D}$ as the Hamiltonian. The operator $\delta\mathfrak{D}$ will account for both weak coupling effects, i.e mixing and anomalous dimension. The Fermi liquid appears when we study eigenstates of this Hamiltonian.

The 1-dim Fermi-sea comes about when one discusses the maximal j_L representation allowed by unitarity [28], i.e., the first line in (1.1). In the parton picture one finds it by restricting the theory to the sector known as the fermionic $\mathfrak{su}(1, 1)$ sector [7, 26]. The restriction is done by using only partons such that

$$(q_1, p, q_2) = (0, 0, q_2)\,, \qquad j_L = j_R + \frac{q_2}{2} \ . \tag{3.5}$$

The only partons obeying this are

$$\psi_{(n)} := \frac{1}{(n+1)!} \left(D_{1\dot{1}}\right)^n \Psi_{1\,4}\,, \tag{3.6}$$

where the normalization is chosen for convinces in later calculations. We will refer to the number of derivatives as the level (n), and the angular momentum (to which we will loosely refer as momentum from now on) is $((n + 1)/2, n/2)$.

Operators with some scaling properties similar to the 1/16 BH - i.e. equation (1.1) - are constructed as follows. Consider a state in this sector with charge q_2 and minimal conformal dimension. The charge dictates that we use q_2 fermionic operators of type (3.6) and for

[5] Derivatives are replaced by covariant derivatives

minimal dimension it best to have as little amount of derivatives as possible. However, as the fermions obey Pauli's exclusion principle we can use each fermion only once (bearing in mind that the quantum numbers of the fermions also include an adjoint gauge index). The 1-dim Fermi-sea operator is then

$$\mathcal{O}^{(K)}_{\text{1dim}} := \prod_{n=0}^{K} \psi^1_{(n)}\psi^2_{(n)}\cdots\psi^{\dim G}_{(n)} := \prod_{n=0}^{K} \text{Jdet}\left(\psi_{(n)}\right) \ . \tag{3.7}$$

The superscripts are the adjoint representation indices, the subscripts encode the level of the parton and in the final equation we introduced the notation Jdet to denotes the product of the all the fermions at the same level. Note that each Jdet is gauge invariant on its own. In effect it is the volume form of the group. All fermionic operator are evaluated at the same space point, correspondingly the expression can be viewed as a state in radial quantization.

The 1-dim Fermi-sea at zero coupling, large N and large K has the following charge, dimension and angular momenta:

$$(q_1, p, q_2) = \sum_{n=0}^{K}\sum_{a=1}^{\dim G} (0,0,1) = (N^2-1)(K+1)\,(0,0,1) \approx \left(0,0,N^2K\right), \tag{3.8a}$$

$$d_0 = \sum_{n=0}^{K}\sum_{a=1}^{\dim G} \left(\frac{3}{2}+n\right) = (N^2-1)\frac{(K+3)(K+1)}{2} \approx \frac{N^2K^2}{2}, \tag{3.8b}$$

$$\begin{aligned}(j_L, j_R) &= \sum_{n=0}^{K}\sum_{a=1}^{\dim G} \left(\frac{n+1}{2}, \frac{n}{2}\right) = (N^2-1)\frac{K(K+1)}{4}\left(\frac{K+2}{K}, 1\right)\\ &\approx \left(\frac{N^2K^2}{4}, \frac{N^2K^2}{4}\right) \ .\end{aligned} \tag{3.8c}$$

Eliminating K from equation (3.8a) and inserting it into equation (3.8c), we obtain the scaling relation (1.1). Note, however, that the coefficient in front of this relation (omitted in (1.1)) does not come out correctly, nor have we identified a large entropy ensemble of operators.

This operator is rather unique in that it does not mix with any other operator in perturbation theory. This is easy to see even without considering the explicit form of the quantum corrected dilatation operator. To do so we consider the effect of weak coupling on (3.7), and check all possible mixing. It is enough to check with which operators $\mathcal{O}^{(K)}_{1-\text{dim}}$ may have a non-vanishing correlator. Since the 1-dim Fermi-sea belongs to the fermionic $\mathfrak{su}(1,1)$ sector, it can only mix with operators in this sector. Due to conservation of the q_2 charge the only possibility is a momentum hoping, i.e

$$(\cdots D_{1\dot{1}}\psi_{1\,4})\,(\cdots \psi_{1\,4}) \leftrightarrow (\cdots \psi_{1\,4})\,(\cdots D_{1\dot{1}}\psi_{1\,4}) \ .$$

However since angular momentum is conserved, the state must be identical to the original, or there will be two fermion with same momentum which is forbidden by Pauli's exclusion principle. Thus the 1-dim Fermi-sea operator is an eigenstates of $\delta\mathfrak{D}$. In the following subsection we will examine this statement in one and two loops.

3.1 The g^2 Hamiltonian

There are a few ways to derive the g^2 Hamiltonian - Feynman diagrams, the harmonic action (discussed in section 2) and algebraic structure. We will use the latter algebraic method since it is easiest to extend it to the g^4 Hamiltonian.

The algebraic method was first introduce by Beisert [7, 26]. The first step is to address the operation of the $\mathfrak{psu}(2,2|4)$ on the fermionic $\mathfrak{su}(1,1)$ sector.[6] This sector is preserved by an $\mathfrak{su}(1,1) \times \mathfrak{u}(1|1)$ subgroup of $\mathfrak{psu}(2,2|4)$, which acts as follows:

- The $\mathfrak{su}(1,1)$ algebra is generated by

$$\mathfrak{J}^0(g) = -\mathcal{L} + 2\mathfrak{D}_0 + \delta\mathfrak{D}(g) \qquad \mathfrak{J}^{++}(g) = \mathfrak{P}_{1\dot{1}} \qquad \mathfrak{J}^{--}(g) = \mathfrak{K}^{\dot{1}1}(g)\,, \tag{3.9a}$$

  where $\mathcal{L}$ is the length (parton number) operator. $\mathcal{L}$ commutes with the entire $\mathfrak{su}(1,1)$ algebra.

- The $\mathfrak{u}(1|1)$ algebra is generated by

$$\begin{aligned} &\mathfrak{T}^+(g) = \bar{\mathfrak{Q}}_{\dot{2}\,4}(g), && \mathcal{L} \\ &\bar{\mathfrak{T}}^-(g) = \bar{\mathfrak{S}}^{\dot{2}\,4}(g), && \delta\mathfrak{D}(g) = 2\left\{\mathfrak{T}^+(g), \bar{\mathfrak{T}}^-(g)\right\}\,. \end{aligned} \tag{3.9b}$$

In the above expressions we made explicit use of the relation between the charges in the sector, for example $\mathfrak{D}_0 = 2\mathfrak{L}^1_1 - \frac{1}{2}\mathcal{L} = 2\bar{\mathfrak{L}}^{\dot{1}}_{\dot{1}} - \frac{3}{2}\mathcal{L}$. Also, for these relations to work it is important to use a regularization scheme (when computing Feynman diagrams) such that the momentum ($\mathfrak{P}$), the lorentz rotations ($\mathfrak{L}$, $\bar{\mathfrak{L}}$) and R-symmetry ($\mathfrak{R}$) receive no quantum correction.

While the $\mathfrak{su}(1,1)$ algebra acts on the $\mathfrak{su}(1,1)$ sector partons at zero coupling, the action of the $\mathfrak{u}(1|1)$ algebra vanishes at g^0, except $\mathcal{L}$. Next we write the generators as

$$\mathfrak{X} = \sum_{n=0}^{\infty} \mathfrak{X}_n\, g^n\,, \qquad \mathfrak{X} \in \mathfrak{su}(1,1) \times \mathfrak{u}(1|1) \tag{3.10}$$

where $g^2 := \frac{g_{\rm ym}^2 N^2}{8\pi^2}$ is the 't Hooft coupling. The generator $\bar{\mathfrak{T}}^{\pm}(g)$ may have a g^1 correction, forcing the first correction to $\delta\mathfrak{D}$ being at g^2. The g^1 terms in $\mathfrak{su}(1|1)$ are determined algebraically as follows: The product structure of $\mathfrak{su}(1,1) \times \mathfrak{u}(1|1)$ remains, leading to the following commutators

$$\begin{aligned} &\left[\mathfrak{J}_0^{++}, \mathfrak{T}_1^+\right] = 0 && \left[\mathfrak{J}_0^{--}, \mathfrak{T}_1^+\right] = 0 && \left[\mathfrak{J}_0^3, \mathfrak{T}_1^+\right] = 0 \\ &\left[\mathfrak{J}_0^{++}, \bar{\mathfrak{T}}_1^-\right] = 0 && \left[\mathfrak{J}_0^{--}, \bar{\mathfrak{T}}_1^-\right] = 0 && \left[\mathfrak{J}_0^3, \bar{\mathfrak{T}}_1^-\right] = 0\,. \end{aligned} \tag{3.11}$$

Considering a spin chain state, the leading order representation of the algebra is

$$\mathfrak{J}_0^{++} \mid \psi_{(n)} \rangle = (n+2) \mid \psi_{(n+1)} \rangle \qquad \mathfrak{J}_0^{--} \mid \psi_{(n)} \rangle = n \mid \psi_{(n)} \rangle \qquad . \tag{3.12}$$

[6] For order g^4 we will need to consider the $\mathfrak{psu}(1,1|2)$ sector, but at the order g^2 we simply quote the results already at the fermionic $\mathfrak{su}(1,1)$ sector.

From the Feynman diagrams expansion (and charges) we know that at order g^1 the generator $\mathfrak{T}^+$ changes a single spin state to a two spin state while $\bar{\mathfrak{T}}^-$ does the opposite. Thus the most general action for the generators consistent with the conserved charges is

$$\mathfrak{T}_1^+ \mid \psi_{(n)} \rangle = \sum_{m=0}^{n-1} a_{n;m} \mid \psi_{(m)}\psi_{(n-1-m)} \rangle \tag{3.13a}$$

$$\bar{\mathfrak{T}}_1^- \mid \psi_{(m)}\psi_{(n)} \rangle = \bar{a}_{n;m} \mid \psi_{n+m+1} \rangle \ . \tag{3.13b}$$

Inserting this equation and (3.12) into (3.11) allows us to determine $a_{m,n}$ and $\bar{a}_{m,n}$ up to an overall factor (which is calculated from comparison to Feynman diagrams). One subtlety in the calculation is that the following state is identified with zero

$$\mid \psi_{(0)}\psi_{(n)} \rangle + \mid \psi_{(n)}\psi_{(0)} \rangle \mathrel{\widehat{=}} 0\,, \tag{3.14}$$

since it is a gauge variation of another expression, and hence will vanish in all gauge invariant expression. This identification is crucial for the existence of a non-trivial solution for the generators. The final result is

$$\mathfrak{T}_1^+ \mid \psi_{(n)} \rangle = \frac{1}{\sqrt{2}} \sum_{m=0}^{n-1} \mid \psi_{(m)}\psi_{(n-1-m)} \rangle \tag{3.15a}$$

$$\bar{\mathfrak{T}}_1^- \mid \psi_{(m)}\psi_{(n)} \rangle = \frac{1}{\sqrt{2}} \left(\frac{1}{n+1} + \frac{1}{m+1} \right) \mid \psi_{n+m+1} \rangle \tag{3.15b}$$

Here we take a different step than [7, 26], and follow ideas presented in [29] - we lift the result to finite N. First we write spin chain states as a long trace

$$\mid X \rangle \to \operatorname{Tr}\left(\cdots (X^a t^a) \cdots\right) \ . \tag{3.16}$$

Then the generators (3.15a)–(3.15b) can be written using matrix operators

$$\mathfrak{T}_1^+ = +\frac{1}{\sqrt{2}} \sum_{k,q=0}^{\infty} \operatorname{Tr} \,{:}\psi_{(k)}\psi_{(q)}\check{\psi}_{(k+q+1)}{:} \tag{3.17a}$$

$$\bar{\mathfrak{T}}_1^- = +\frac{1}{\sqrt{2}}\frac{1}{N} \sum_{m,n=0}^{\infty} \left(\frac{1}{n+1} + \frac{1}{m+1} \right) \operatorname{Tr} \,{:}\psi_{(n+m+1)}\check{\psi}_{(n)}\check{\psi}_{(m)}{:} \tag{3.17b}$$

$\check{\psi}_{(n)}$ is a subset of the $\check{\mathcal{W}}$ introduced before (up to a different normalization in (3.6)) - it removes an operator $\psi_{(n)}$ and satisfies $\left\{ \check{\psi}^a_{(n)}, \psi^b_{(m)} \right\} := \delta_{nm}\delta^{ab}$ where a, b are adjoint indices of the gauge group. The colon's stands for normal ordering i.e

$${:}\check{\psi}^a_{(n)}\psi^b_{(m)}{:} := \check{\psi}_{(n)}\psi_{(m)} - \delta_{nm}\delta^{ab} = -\psi_{(m)}\check{\psi}_{(n)} \ .$$

Finally, the following $\mathrm{SU}(N)$ relation are useful

$$\sum_a \operatorname{Tr} At^a \operatorname{Tr} Bt^a = \operatorname{Tr} AB - \frac{1}{N} \operatorname{Tr} A \operatorname{Tr} B \tag{3.18a}$$

$$\sum_a \operatorname{Tr} At^a Bt^a = \operatorname{Tr} A \operatorname{Tr} B - \frac{1}{N} \operatorname{Tr} AB \ . \tag{3.18b}$$

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We now have the tools to calculate the g^2 correction to $\delta\mathfrak{D}$ at finite N -

$$
\begin{aligned}
\delta\mathfrak{D}_2 =& 2\left\{\mathfrak{T}_1^+, \bar{\mathfrak{T}}_1^-\right\} = \\
=& +\frac{1}{N}\sum_{m,n,k,q=0}^{\infty}\delta_{m+n=k+q}\left(\frac{\Theta(k-m)}{k-m}-\frac{\Theta(q-n)}{m+1}\right)\times \\
& \qquad \times \operatorname{Tr} :\left\{\psi_{(q)},\check{\psi}_{(n)}\right\}\left\{\psi_{(k)},\check{\psi}_{(m)}\right\}: + \\
& -\frac{1}{N}\sum_{m,n=0}^{\infty}\frac{1}{m+1}\operatorname{Tr} :\left\{\psi_{(m)},\check{\psi}_{(m)}\right\}\left\{\psi_{(n)},\check{\psi}_{(n)}\right\}: + \\
& +\sum_{m=0}^{\infty}2h(m+1)\operatorname{Tr} :\psi_{(m+1)}\check{\psi}_{(m+1)}: ,
\end{aligned}
\tag{3.19}
$$

where the Θ are step functions

$$
\Theta(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}
$$

and $h(n)$ stands for the harmonic numbers

$$
h(n) := \sum_{k=1}^{n}\frac{1}{k}, \qquad h(0) := 0 .
$$

This result agrees with the results of [7, 26] if we note that

$$
\operatorname{Tr} :\psi_{(m)}\check{\psi}_{(m)}: = -\frac{1}{2N}\sum_{n=0}^{\infty}\operatorname{Tr} :\left\{\psi_{(m)},\check{\psi}_{(m)}\right\}\left\{\psi_{(n)},\check{\psi}_{(n)}\right\}: \tag{3.20}
$$

since the difference between these operators is a gauge transformation (see appendix B), and we may use (3.20) as an identity when acting on gauge invariant operators. Using this property and some algebra, the dilatation operator up to order g^2 is

$$
\begin{aligned}
\mathfrak{D} =& \sum_{m=0}^{\infty}\left(\frac{3}{2}+m\right)\operatorname{Tr} :\psi_{(m)}\check{\psi}_{(m)}: + g^2\left[\sum_{m=0}^{\infty}2h(m+1)\operatorname{Tr} :\psi_{(m)}\check{\psi}_{(m)}: + \right. \\
& \left. +\frac{1}{N}\sum_{m,k,q=0}^{\infty}\frac{(q+1)\Theta(k-m)}{(k-m)(k+q-m+1)}\operatorname{Tr} :\left\{\psi_{(k)},\check{\psi}_{(m)}\right\}\left\{\psi_{(q)},\check{\psi}_{(k+q-m)}\right\}:\right] + \\
& + O(g^4) .
\end{aligned}
\tag{3.21}
$$

The dilatation operator exhibit the form expected for a Fermi liquid, the leading order in perturbation theory correct the mass terms and add a 2-2 fermion interaction - in section 4 we will discuss the analogy in more details. For now we will calculate the dilatation charge of the Fermi-sea state. The new ingredient in the calculation is the 4-fermions coupling term. The annihilation operators $\check{\psi}_{(m)}$ and $\check{\psi}_{(k+q-m)}$ puncture two holes in the Fermi-sea, thus $m \leq K$ and $k+q-m \leq K$. Since the interaction preserves momentum, at least one fermion creation operator must be below (or equal to) the Fermi-level, but then Pauli's exclusion principle forces it to fill one of the hole created, and we find that the other creation operator must fill the other hole. To summarize we have

$$
\begin{aligned}
\operatorname{Tr} :\left\{\psi_{(k)}, \check{\psi}_{(m)}\right\}\left\{\psi_{(q)}, \check{\psi}_{(k+q-m)}\right\}: & \left| \ \mathcal{O}_{1\mathrm{dim}}^{(K)} \ \right\rangle \\
= & -\sum_{a,b=0}^{\dim G} \delta_{m=q} \operatorname{Tr}\left[t^a, t^b\right]\left[t^b, t^a\right] \left| \ \mathcal{O}_{1\mathrm{dim}}^{(K)} \ \right\rangle = \\
= & -2N(N^2-1)\delta_{q=m} \left| \ \mathcal{O}_{1\mathrm{dim}}^{(K)} \ \right\rangle ,
\end{aligned}
\tag{3.22}
$$

and for the dilatation operator we find

$$
\begin{aligned}
\mathfrak{D} \left| \ \mathcal{O}_{1\mathrm{dim}}^{(K)} \ \right\rangle = & (N^2-1) \sum_{m=0}^{K} \left[\left(\frac{3}{2}+m\right) + g^2\, 2h(m+1) + \right. \\
& \left. - g^2 \sum_{k=m+1}^{K} \frac{2(m+1)}{(k-m)(k+1)} + O(g^4) \right] \left| \ \mathcal{O}_{1\mathrm{dim}}^{(K)} \ \right\rangle = \\
= & (N^2-1) \frac{(K+3)(K+1)}{2} \left[1 + \frac{4\,g^2}{K+3} + O(g^4) \right] \left| \ \mathcal{O}_{1\mathrm{dim}}^{(K)} \ \right\rangle .
\end{aligned}
\tag{3.23}
$$

Note that the g^2 correction is suppressed by $1/K$ compared to the leading order - this is the main point of this section. We will see that this is also the case to order g^4. We would like to make two comments that highlight the significance of this result:

- We can compare this result to other operators which are familiar from the spin-chain and integrability literature. For example, consider another eigenstate of $\mathfrak{D}$, which is the analog of the bosonic twist-2 operator (see [7] for details)

$$
\begin{aligned}
\mathcal{O}_{\mathrm{twist2}} := & \sum_{n=0}^{K} \frac{(-)^n (K)!}{n!(K-n)!} \operatorname{Tr}\left(\psi_{(n)}\psi_{(K-n)}\right) \\
\mathfrak{D} \mid \mathcal{O}_{\mathrm{twist2}} \ \rangle = & (K+3)\left[1 + g^2 \frac{2h(K+1)}{K+3}\right] \mid \mathcal{O}_{\mathrm{twist2}} \ \rangle
\end{aligned}
\tag{3.24}
$$

  Using the asymptotics $h(K) \sim \log K$ for large K, we see that the order g^2 ratio of the anomalous dimension over the classical dimension scales, for large K, as $\sim (\log K)/K$, whereas in our case the g^2 correction is of order $1/K$.

- Consider an operator in which we fill fermions states in a sparse way up to some maximal level K. As long as the filling is sparse such an operator can be written as a product of (3.24) operators. Since the g^2 anomalous dimension is a two-to-two parton operator, the $\sim (\log K)/K$ will persist for such sparse operators. Cancelation starts occurring as the Fermi shells become more and more filled and Fermi statistics becomes the dominant feature in the form of the operator.

- We will see later that the cancelation of the logarithms persists to order g^4, i.e., terms of the form $\sim N^2 g^4 K (\log K)^2$ and $\sim N^2 g^4 K \log K$ do not appear in the anomalous dimension. The cancelation of the first set of terms, along with the cancelation of $\sim N^2 g^2 K \log K$ at this order, is important. Had these terms been there, it would have suggested that the correction to the dimension is of the form $\sim N^2 K^{1+F(g^2)}$ which can become dominant at large g^2. In our case, the cancelation of these terms suggests that the correction to the dimension is of the form $\sim N^2 K F(g^2)$ which is parametrically smaller than the classical value of $N^2 K^2$ even for large g^2.

3.2 The g^4 Hamiltonian

An algebraic solution for the order g^4 dilatation operator, based on the $\mathfrak{psu}(1,1|2)$ sector, was found by Zwiebel [29]. This sector is generated by the partons

$$\begin{aligned} \psi_{(n)} &:= \frac{1}{(n+1)!} (D_{1\dot{1}})^n \Psi_{14} & \phi^2_{(n)} &:= \frac{1}{(n)!} (D_{1\dot{1}})^n \Phi_{24} \\ \bar{\psi}_{(n)} &:= \frac{1}{(n+1)!} (D_{1\dot{1}})^n \bar{\Psi}^1_{\dot{1}} & \phi^3_{(n)} &:= \frac{1}{(n)!} (D_{1\dot{1}})^n \Phi_{34} \, . \end{aligned} \tag{3.25}$$

Zwiebel's solution is reviewed in appendix C. Here we quote the final result,

$$\delta\mathfrak{D}_4 = +2\left\{\bar{\mathfrak{T}}^-_1, \left[\mathfrak{T}^+_1, \left\{\mathfrak{T}^-_1, \left[\bar{\mathfrak{T}}^+_1, \mathfrak{h}\right]\right\}\right]\right\} + 2\left\{\mathfrak{T}^+_1, \left[\bar{\mathfrak{T}}^-_1, \left\{\bar{\mathfrak{T}}^+_1, \left[\mathfrak{T}^-_1, \mathfrak{h}\right]\right\}\right]\right\} , \tag{3.26}$$

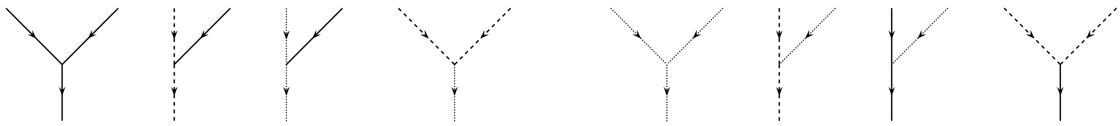
with

$$\begin{aligned} \mathfrak{h} :=& \sum_{n=0}^{\infty} +\frac{1}{2} h(n+1) \,\mathrm{Tr} :\psi_{(n)} \check{\psi}_{(n)}: + \sum_{n=0}^{\infty} \frac{1}{2} h(n+1) \,\mathrm{Tr} :\bar{\psi}_{(n)} \check{\bar{\psi}}_{(n)}: + \\ &+ \sum_{n=0}^{\infty} \frac{1}{2} h(n) \sum_{i=2}^{3} \mathrm{Tr} :\phi^i_{(n)} \check{\phi}^i_{(n)}: , \end{aligned} \tag{3.27}$$

and $\mathfrak{T}^\pm_1$ and $\bar{\mathfrak{T}}^\pm_1$ are the leading order (g^1) supercharges of the $\mathfrak{psu}(1|1)^2$ algebra, which commutes with the $\mathfrak{psu}(1,1|2)$ algebra (which can be found by a generalization of the calculation for $\mathfrak{su}(1,1)$ fermionic sector)

$$\begin{aligned} \mathfrak{T}^+_1 =& \frac{1}{\sqrt{2}} \sum_{k,q=0}^{\infty} \Bigg(\mathrm{Tr} :\psi_{(k)} \psi_{(q)} \check{\psi}_{(k+q+1)}: + \sum_{i=2}^{3} \mathrm{Tr} : \left[\psi_{(k)}, \phi^i_{(q)}\right] \check{\phi}^i_{(k+q+1)}: + \\ &+ \frac{q+1}{(k+q+2)} \mathrm{Tr} : \left\{\psi_{(k)}, \bar{\psi}_{(q)}\right\} \check{\bar{\psi}}_{(k+q+1)}: + \\ &- \frac{1}{k+q+1} \mathrm{Tr} : \left[\phi^2_{(k)}, \phi^3_{(q)}\right] \check{\bar{\psi}}_{(k+q)}: \Bigg) \end{aligned} \tag{3.28a}$$

$$\begin{aligned} \bar{\mathfrak{T}}^+_1 =& \frac{1}{\sqrt{2}} \sum_{k,q=0}^{\infty} \Bigg(\mathrm{Tr} :\bar{\psi}_{(k)} \bar{\psi}_{(q)} \check{\bar{\psi}}_{(k+q+1)}: + \sum_{i=2}^{3} \mathrm{Tr} : \left[\bar{\psi}_{(k)}, \phi^i_{(q)}\right] \check{\phi}^i_{(k+q+1)}: + \\ &+ \frac{q+1}{(k+q+2)} \mathrm{Tr} : \left\{\bar{\psi}_{(k)}, \psi_{(q)}\right\} \check{\psi}_{(k+q+1)}: + \\ &+ \frac{1}{k+q+1} \mathrm{Tr} : \left[\phi^2_{(k)}, \phi^3_{(q)}\right] \check{\psi}_{(k+q)}: \Bigg) \end{aligned} \tag{3.28b}$$



(a) The $\bar{\mathfrak{Q}}_1^+$ vertices (b) The $\bar{\mathfrak{Q}}_1^-$ vertices

Figure 1: The continuous, dashed, doted lines stands for $\psi_{(n)}$, $\phi^i_{(n)}$ and $\bar{\psi}_{(n)}$ respectively. The arrows indicate the flow of momentum. For the conjugated diagrams $\bar{\mathfrak{Q}}_1^-$ and $\mathfrak{Q}_1^+$ the vertices structure do not change, only the arrows changes sign. Note that these are not Feynman diagrams, but rather they only describe the how the various terms in these operators act.

$$
\begin{aligned}
\mathfrak{Q}_1^- =& \frac{1}{\sqrt{2}N} \sum_{m,n=0}^{\infty} \Bigg(\frac{n+m+2}{(n+1)(m+1)} \operatorname{Tr} :\bar{\psi}_{(n+m+1)} \check{\bar{\psi}}_{(n)} \check{\bar{\psi}}_{(m)}: + \\
& - \operatorname{Tr} :\psi_{(n+m)} \left[\check{\phi}^2_{(n)}, \check{\phi}^3_{(m)}\right]: + \frac{1}{n+1} \operatorname{Tr} :\psi_{(n+m+1)} \left\{\check{\bar{\psi}}_{(n)}, \check{\psi}_{(m)}\right\}: + \\
& + \frac{1}{n+1} \sum_{i=2}^{3} \operatorname{Tr} :\phi^i_{(n+m+1)} \left[\check{\bar{\psi}}_{(n)}, \check{\phi}^i_{(m)}\right]: \Bigg)
\end{aligned} \tag{3.28c}
$$

$$
\begin{aligned}
\bar{\mathfrak{Q}}_1^- =& \frac{1}{\sqrt{2}N} \sum_{m,n=0}^{\infty} \Bigg(\frac{n+m+2}{(n+1)(m+1)} \operatorname{Tr} :\psi_{(n+m+1)} \check{\psi}_{(n)} \check{\psi}_{(m)}: + \\
& + \operatorname{Tr} :\bar{\psi}_{(n+m)} \left[\check{\phi}^2_{(n)}, \check{\phi}^3_{(m)}\right]: + \frac{1}{n+1} \operatorname{Tr} :\bar{\psi}_{(n+m+1)} \left\{\check{\psi}_{(n)}, \check{\bar{\psi}}_{(m)}\right\}: + \\
& + \frac{1}{n+1} \sum_{i=2}^{3} \operatorname{Tr} :\phi^i_{(n+m+1)} \left[\check{\psi}_{(n)}, \check{\phi}^i_{(m)}\right]: \Bigg) .
\end{aligned} \tag{3.28d}
$$

There are two caveats in Zwiebel's solutions (already discussed in [29]): (1) It is not clear that the solution is unique. In general there could exist a "homogenous" set of deformations of the generators (beside the coupling redefinition and similarity transformation) which do not effect the commutation relations. No such "homogeneous solutions" are known though. (2) The lift to the non-planarity is not proven to be unique. Some checks at large N (by comparison to Feynman diagrams) of the solution for small spin chain operators are provided in [29]. The appearance of a Yangian symmetry [30] also supports the solution, at least in the spin chain limit. For more recent discussion about the validity of the solution beyond the planar limit see [31].

The supercharges $\mathfrak{Q}_1^{\pm}$ and $\bar{\mathfrak{Q}}_1^{\pm}$ can be described diagrammatically as a set of vertices (see figure 1) - the $\mathfrak{Q}_1^{\pm}$ split a single parton into a pair of partons and the $\bar{\mathfrak{Q}}_1^{\pm}$ combine a pair of parton into a single parton. For our purpose, it is enough to focus on the fermionic $\mathfrak{su}(1,1)$ sector, and examine how each operator acts on the sector.

First consider the contribution of the operators $\mathfrak{Q}_1^-$, $\bar{\mathfrak{Q}}_1^+$ in (3.26). Since $\mathfrak{h}$ is a 1-1 parton operator, which cannot change the parton, the combinations $\left\{\mathfrak{Q}_1^-, \left[\bar{\mathfrak{Q}}_1^+, \mathfrak{h}\right]\right\}$ and $\left\{\bar{\mathfrak{Q}}_1^+, \left[\mathfrak{Q}_1^-, \mathfrak{h}\right]\right\}$ have the same structure as $\delta\mathfrak{D}_2$ (the difference is in coefficients only). Since $\delta\mathfrak{D}_2$ acts within the fermionic $\mathfrak{su}(1,1)$ sector, these combinations acts within the fermionic $\mathfrak{su}(1,1)$ sector too. The operators $\mathfrak{Q}_1^+$, $\bar{\mathfrak{Q}}_1^-$ and $\delta\mathfrak{D}_2$ also act within the fermionic $\mathfrak{su}(1,1)$

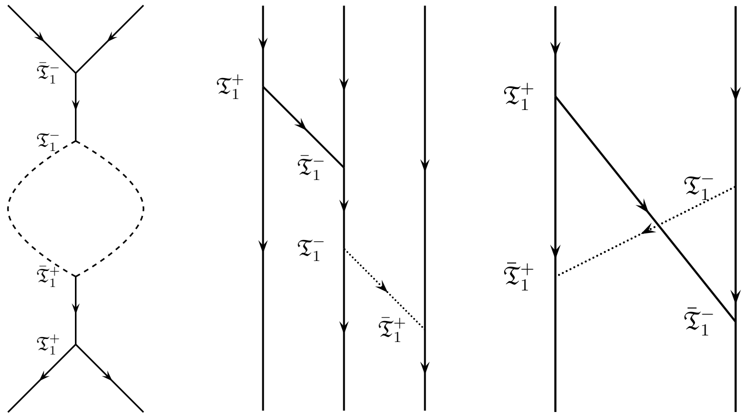


Figure 2: Three typical diagrams for $\delta\mathfrak{D}_4$, which keep track of how the partons change when acting on them with the sequence of operators in (3.26). The vertices are ordered in the vertical axis by the order the operators acts on the state. Since $\mathfrak{T}^-$, $\bar{\mathfrak{T}}_1^+$ appear in the inner commutators only, the corresponding vertices are inserted sequently, as a result they form a either a 1-1 and 2-2 fermionic operator, any none fermionic line emerging from the inner commutator will continue to the outer legs (irrelevant for our purpose).

sector. Hence, after we calculate the inner commutators (involving $\mathfrak{T}_1^-$, $\bar{\mathfrak{T}}_1^+$ and $\mathfrak{h}$) in the full $\mathfrak{psu}(1,1|2)$ sector we can continue the calculation in the smaller $\mathfrak{su}(1,1)$ sector. Some sample diagrams, contributing to $\delta\mathfrak{D}_4$, which illustrate the argument are sketched in figure 2 (these are not Feynman diagrams. These diagrams only keep track of the partons and of the ordering of how the operators act on them).

We first evaluate $\delta\mathfrak{D}_4$ on the $\mathfrak{su}(1,1)$ sector, and then use the resulting expression to evaluate it on $\left| \mathcal{O}^{(K)}_{1\text{dim}} \right\rangle$. First we take care of the inner commutators

$$2\left\{\bar{\mathfrak{T}}_1^+, \left[\mathfrak{T}_1^-, \mathfrak{h}\right]\right\} = \mathfrak{U} - \mathfrak{G} \tag{3.29a}$$

$$2\left\{\mathfrak{T}_1^-, \left[\bar{\mathfrak{T}}_1^+, \mathfrak{h}\right]\right\} = \mathfrak{G} - \mathfrak{V}\,, \tag{3.29b}$$

with

$$\mathfrak{U} = \frac{1}{2N}\sum_{\substack{m,n,\\k,q=0}}^{\infty} \delta_{m+n=k+q}\mathcal{A}_{(m,n-q-1)}\mathcal{B}_{[n-q-1,n]}\,\mathrm{Tr}\,:\left\{\psi_{(k)},\check{\psi}_{(m)}\right\}\left\{\psi_{(q)},\check{\psi}_{(n)}\right\}: \tag{3.30a}$$

$$\mathfrak{V} = \frac{1}{2N}\sum_{\substack{m,n,\\k,q=0}}^{\infty} \delta_{m+n=k+q}\mathcal{A}_{(q,n-q-1)}\mathcal{B}_{[n-q-1,n]}\,\mathrm{Tr}\,:\left\{\psi_{(k)},\check{\psi}_{(m)}\right\}\left\{\psi_{(q)},\check{\psi}_{(n)}\right\}: \tag{3.30b}$$

$$\mathfrak{G} = -\frac{1}{2}\sum_{m=0}^{\infty}\mathcal{D}_{(m)}\,\mathrm{Tr}\,:\psi_{(m)}\check{\psi}_{(m)}:\,, \tag{3.30c}$$

where the constants are

$$\mathcal{A}_{(a,b)} = \frac{h(a+1)+h(b+1)-h(a+b+2)}{2}\Theta(a+1)\Theta(b+1) \tag{3.31a}$$

$$\mathcal{B}_{[a,b]} = \frac{b-a}{(a+1)(b+1)}\Theta(a+1)\Theta(b+1) \tag{3.31b}$$

$$\mathcal{C}_{(a,b)} = \frac{a+b+2}{(a+1)(b+1)}\Theta(a+1)\Theta(b+1) \tag{3.31c}$$

$$\mathcal{D}_{(a)} = -\frac{h(a+1)}{(a+1)} + \Theta(a)\sum_{b=0}^{a-1}\frac{h(b+1)+h(a-b)-h(a+1)}{a-b} \tag{3.31d}$$

Next we calculate the g^3 terms

$$\left[\mathfrak{T}_1^+, \mathfrak{G}\right] = \frac{1}{4\sqrt{2}}\sum_{k,q=0}^{\infty} \xi_{(k,q)} \ \mathrm{Tr} : \left\{\psi_{(k)}, \psi_{(q)}\right\} \check{\psi}_{(k+q+1)} : \tag{3.32a}$$

$$\left[\bar{\mathfrak{T}}_1^-, \mathfrak{G}\right] = -\frac{1}{4\sqrt{2}\,N}\sum_{k,q=0}^{\infty} \mathcal{C}_{(k,q)}\xi_{(k,q)} \ \mathrm{Tr} : \psi_{(k+q+1)} \left\{\check{\psi}_{(k)}, \check{\psi}_{(q)}\right\} : \tag{3.32b}$$

$$\begin{aligned}\left[\mathfrak{T}_1^+, \mathfrak{V}\right] = &\frac{1}{4\sqrt{2}\,N}\sum_{\substack{m,n,\\ q,k=0}}^{\infty} \nu_{[(q;n),(k;m)]}\times \\ &\times \mathrm{Tr} : \left[\left\{\psi_{(q)}, \check{\psi}_{(n)}\right\}, \left\{\psi_{(k)}, \check{\psi}_{(m)}\right\}\right] \psi_{(m+n-k-q-1)} : + \\ &+ \frac{1}{4\sqrt{2}}\sum_{k,q=0}^{\infty} \bar{\nu}_{(k,q)} \ \mathrm{Tr} : \left\{\psi_{(q)}, \psi_{(k)}\right\} \check{\psi}_{(k+q+1)} :\end{aligned} \tag{3.32c}$$

$$\begin{aligned}\left[\bar{\mathfrak{T}}_1^-, \mathfrak{U}\right] = &\frac{1}{4\sqrt{2}\,N^2}\sum_{\substack{m,n,\\ k,q=0}}^{\infty} \mu_{[(q;n),(k;m)]}\times \\ &\times \mathrm{Tr} : \left[\left\{\psi_{(q)}, \check{\psi}_{(n)}\right\}, \left\{\psi_{(k)}, \check{\psi}_{(m)}\right\}\right] \check{\psi}_{(k+q-m-n-1)} : + \\ &- \frac{1}{4\sqrt{2}\,N}\sum_{k,q=0}^{\infty} \bar{\mu}_{(k,q)} \ \mathrm{Tr} : \psi_{(k+q+1)} \left\{\check{\psi}_{(k)}, \check{\psi}_{(q)}\right\} :\end{aligned} \tag{3.32d}$$

In the above calculation we use the following identity proved in appendix B

$$\begin{aligned}\mathrm{Tr} : \left\{\psi_{(k)}, \psi_{(q)}\right\} \check{\psi}_{(k+q+1)} : &\ \widehat{=} \\ &\widehat{=} \ -\frac{1}{2N}\sum_{m=0}^{\infty}\mathrm{Tr} : \left\{\psi_{(m)}, \check{\psi}_{(m)}\right\} \left[\left\{\psi_{(q)}, \psi_{(k)}\right\}, \check{\psi}_{(k+q+1)}\right] :\end{aligned} \tag{3.33}$$

and define

$$\begin{aligned}\mu_{[(q;n)(k;m)]} := &\Theta(k+q-m-n)\times \\ &\times \Big[A_{(m,k-m-1)}B_{[k+q-m-n-1,k-m-1]}C_{[n-q-1,n]} + \\ &- A_{(k+q-n,n-q-1)}B_{[k+q-m-n-1,k-m-1]}C_{[n-q-1,n]} + \\ &- A_{(n,q-n-1)}B_{[k+q-m-n-1,q-n-1,]}C_{[m-k-1,m]} + \\ &- A_{(k+q-m,m-k-1)}B_{[k+q-m-n-1,q-n-1,]}C_{[m-k-1,m]}\Big]\end{aligned} \tag{3.34a}$$

$$\nu_{[(q;n)(k;m)]} := \Theta(m+n-k-q)\times$$

$$\times\Big(A_{(m+n-k,k-m-1)}B_{[k-m-1,n]}-A_{(m+n-k-q-1,k-m-1)}B_{[k-m-1,n-q-1]}+$$
$$-A_{(m+n-q,q-n-1)}B_{[q-n-1,m]}+A_{(m+n-k-q-1,q-n-1)}B_{[q-n-1,m-k-1]}+$$
$$-\Theta(k-m+1)A_{(q,n-q-1)}B_{[n-q-1,n]}+$$
$$+\Theta(q-n+1)A_{(k,m-k-1)}B_{[m-k-1,m]}\Big) \tag{3.34b}$$

$$\bar{\nu}_{(k,q)}:=\sum_{n=0}^{k+q}\left(\mathcal{A}_{(q,n-q-1)}\mathcal{B}_{[n-q-1,n]}+\mathcal{A}_{(k,n-k-1)}\mathcal{B}_{[n-k-1,n]}\right) \tag{3.34c}$$

$$\bar{\mu}_{(k,q)}:=\sum_{n=0}^{k+q}\left(\mathcal{A}_{(q,k-n-1)}\mathcal{B}_{[k-n-1,k]}+\mathcal{A}_{(k,q-n-1)}\mathcal{B}_{[q-n-1,q]}\right)\mathcal{C}_{(k+q-n,n)} \tag{3.34d}$$

$$\Gamma(k,q):=\mathcal{D}_{(k)}+\mathcal{D}_{(q)}-\mathcal{D}_{(k+q+1)} \tag{3.34e}$$

The notations in the above are such that the brackets (a,b) indicates a symmetry replacing the momenta between them, the square brackets $[a,b]$ indicates antisymmetry, and brackets with a semicolon $(a;b)$ has no symmetry. The last stage is calculating $\delta\mathfrak{D}_4$, for which we can use

$$\delta\mathfrak{D}_4=\left\{\bar{\mathfrak{T}}_1^-,\left[\mathfrak{T}_1^+,\mathfrak{G}\right]\right\}-\left\{\bar{\mathfrak{T}}_1^-,\left[\mathfrak{T}_1^+,\mathfrak{V}\right]\right\}-\left\{\mathfrak{T}_1^+,\left[\bar{\mathfrak{T}}_1^-,\mathfrak{G}\right]\right\}+\left\{\mathfrak{T}_1^+,\left[\bar{\mathfrak{T}}_1^-,\mathfrak{U}\right]\right\}\ . \tag{3.35}$$

Summing all the contributions we find

$$\delta\mathfrak{D}_4=+\frac{1}{4N^2}\sum_{\substack{m,n,k,q,\\ r,s,t,l=0}}^{\infty}\mu_{[(q;n),(k;m)]}\delta_{l=k+q-m-n-1}\delta_{t=r+s+1}\times$$
$$\times\Bigg[-\delta_{s=n}\operatorname{Tr}:\left[\left\{\psi_{(k)},\check{\psi}_{(m)}\right\},\check{\psi}_{(l)}\right]\left[\left\{\psi_{(r)},\check{\psi}_{(t)}\right\},\psi_{(q)}\right]:+$$
$$+\delta_{t=q}\operatorname{Tr}:\left[\left\{\psi_{(k)},\check{\psi}_{(m)}\right\},\check{\psi}_{(l)}\right]\left[\left\{\psi_{(r)},\check{\psi}_{(n)}\right\},\psi_s\right]:+$$
$$-\delta_{r=l}\operatorname{Tr}:\left[\left\{\psi_{(k)},\check{\psi}_{(m)}\right\},\check{\psi}_{(t)}\right]\left[\left\{\psi_{(q)},\check{\psi}_{(n)}\right\},\psi_{(s)}\right]:+$$
$$-\delta_{s=n}\delta_{r=l}N\operatorname{Tr}:\left\{\psi_{(k)},\check{\psi}_{(m)}\right\}\left\{\psi_{(q)},\check{\psi}_{(t)}\right\}:+$$
$$+\delta_{s=n}\delta_{r=m}\Big(N\operatorname{Tr}:\psi_{(q)}\check{\psi}_{(t)}\psi_{(k)}\check{\psi}_{(l)}:+2\operatorname{Tr}:\psi_{(q)}\check{\psi}_{(t)}:\operatorname{Tr}:\psi_{(k)}\check{\psi}_{(l)}:+$$
$$+\operatorname{Tr}:\psi_{(q)}\psi_{(k)}:\operatorname{Tr}:\check{\psi}_{(l)}\check{\psi}_{(t)}:\Big)\Bigg]+$$
$$+\frac{1}{4N^2}\sum_{\substack{m,n,k,q,\\ r,s,l,t=0}}^{\infty}C_{(r,s)}\nu_{[(q;n),(k;m)]}\delta_{l=m+n-k-q-1}\delta_{t=r+s+1}\times$$
$$\times\Bigg[-\delta_{s=q}\operatorname{Tr}:\left[\left\{\psi_{(t)},\check{\psi}_{(r)}\right\},\check{\psi}_{(n)}\right]\left[\left\{\psi_{(k)},\check{\psi}_{(m)}\right\},\psi_{(l)}\right]:+$$
$$+\delta_{t=n}\operatorname{Tr}:\left[\left\{\psi_{(q)},\check{\psi}_{(r)}\right\},\check{\psi}_{(s)}\right]\left[\left\{\psi_{(k)},\check{\psi}_{(m)}\right\},\psi_{(l)}\right]:+$$
$$-\delta_{r=l}\operatorname{Tr}:\left[\left\{\psi_{(q)},\check{\psi}_{(n)}\right\},\check{\psi}_{(s)}\right]\left[\left\{\psi_{(k)},\check{\psi}_{(m)}\right\},\psi_{(t)}\right]:+$$
$$-\delta_{s=q}\delta_{r=l}N\operatorname{Tr}:\left\{\psi_{(t)},\check{\psi}_{(n)}\right\}\left\{\psi_{(k)},\check{\psi}_{(m)}\right\}:+$$

$$
\begin{aligned}
&+\delta_{s=q}\delta_{r=k}\Big(+N\,\mathrm{Tr}:\psi_{(t)}\check{\psi}_{(n)}\psi_{(l)}\check{\psi}_{(m)}:+2\,\mathrm{Tr}:\psi_{(t)}\check{\psi}_{(n)}:\mathrm{Tr}:\psi_{(l)}\check{\psi}_{(m)}:+\\
&\qquad\qquad+\mathrm{Tr}:\psi_{(t)}\psi_{(l)}:\mathrm{Tr}:\check{\psi}_{(m)}\check{\psi}_{(n)}:\Big)\Big]+\\
&+\frac{1}{2N}\sum_{r,s,k,q=0}^{\infty}\big(C_{(k,q)}\xi_{(r,s)}+C_{(k,q)}\xi_{(k,q)}-\bar{\mu}_{(k,q)}-C_{(k,q)}\bar{\nu}_{(r,s)}\big)\times\\
&\times\Big(-\frac{1}{2}\delta_{k+q=r+s}\,\mathrm{Tr}:\big\{\psi_{(s)},\check{\psi}_{(k)}\big\}\big\{\psi_{(r)},\check{\psi}_{(q)}\big\}:+\\
&\quad+\delta_{r=k}\,\mathrm{Tr}:\big\{\psi_{(s)},\check{\psi}_{(k+s+1)}\big\}\big\{\psi_{(k+q+1)},\check{\psi}_{(q)}\big\}:+\\
&\quad+\delta_{s=k}\delta_{r=q}N\,\mathrm{Tr}:\psi_{(k+q+1)}\check{\psi}_{(k+q+1)}:\Big)
\end{aligned}\tag{3.36}
$$

We are set to apply the dilatation operator to the Fermi-sea. The details are given in appendix D, and here we only quote the result in the large K and large N limit

$$
\mathfrak{D}\Big|\,\mathcal{O}^{(K)}_{\mathrm{1dim}}\Big\rangle=N^2\frac{K^2}{2}\left[1+\frac{4\,g^2}{K}-\frac{4g^4}{K}+O(K^{-2})+O(g^6)\right]\Big|\,\mathcal{O}^{(K)}_{\mathrm{1dim}}\Big\rangle\;. \tag{3.37}
$$

This is the main result of the paper, as far as the explicit computation of the dimension is concerned.

Before proceeding to analyze the result it is important to emphasize that the expression in (3.37) contains contributions from non-planar diagrams which are not suppressed by power of N compared to the planar ones. This, however, is not in contradiction to the standard lore, since the operator has a strong N dependence as it has order N^2K partons (stronger in fact than determinants/giant gravitons [32, 33]).

We can also repeat the analysis that we had before, examining which terms cancel, and compare to the bosonic twist operator. At order g^4 there are $K(\log K)^2$ and $K\log K$ coefficients which vanish (the significance of this was discussed after equation (3.23)). Compared to the bosonic twist 2 operator, the latter has a leading log behavior

$$
\begin{aligned}
&\mathfrak{D}|\ \mathrm{Tr}\,(\Phi_{14}(D_{1\dot{1}})^s\Phi_{14})\ \rangle=\\
&\qquad=s\left[1+f(g)\frac{\log s}{s}+h(g)\frac{\log^2 s}{s}+O(s^0)\right]|\ \mathrm{Tr}(\Phi_{14}(D_{1\dot{1}})^sM_{14})\ \rangle
\end{aligned}\tag{3.38}
$$

with $f(g)=4g^2-\frac{2}{3}\pi^2g^4+\cdots$ (replacing the scalar in the above with a fermion will not make a considerable change of the result). As before, the situation for the Fermi-sea is better than the expected expansion in $g^{2n}(\log K)^n$. Rather, the persistence of the cancelation to order g^4 suggests that the behavior at large K is

$$
\mathfrak{D}\Big|\,\mathcal{O}^{(K)}_{\mathrm{1dim}}\Big\rangle=N^2\frac{K^2}{2}\left[1+\frac{F(g^2)}{K}+O(K^{-2})\right]\Big|\,\mathcal{O}^{(K)}_{\mathrm{1dim}}\Big\rangle\;. \tag{3.39}
$$

The function $F(g)$ is analogous to the famous cusp anomalous dimension calculated to all order by Beisert, Eden and Staudacher [9, 34] for the twist 2 operators.[7] From the work

[7] See also [10] for a generalization to twist-j operators.

of Frolov and Tseytlin [14], we know that the anomalous dimension of a twist 2 operator remains finite at strong coupling ($g \to \infty$), as it is dual to a folded spinning string [13]. As the situation for the Fermi-sea operator seems somewhat better - in the sense that the corrections are relatively smaller - one can hope that this class of operators continues into to strong coupling, where they will be 1/16 BPS states, up to small corrections.

4. Quasi particles

Next, we study the quasi-particles - the small excitations above the Fermi surface - to order g^2. We will not compute all excitations of the Fermi surface by all operators in the CFT, but rather only excitations in the fermionic $\mathfrak{su}(1,1)$ sector. The starting point is the free-energy restricted to this sector

$$\mathcal{F} = \delta\mathfrak{D} = g^2 \sum_{k=0}^{\infty} \Bigg[2h(k+1) \sum_a \psi^a_{(k)} \check{\psi}^a_{(k)} + \\ + \frac{1}{N} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{a,b,c,d,e} \frac{(p+1)\Theta(k-p+q)}{(k-p+q)(k+q+1)} f^e_{ab} f^e_{cd} \psi^a_{(k)} \psi^c_{(p)} \check{\psi}^b_{(p-q)} \check{\psi}^d_{(k+q)} \Bigg], \quad (4.1)$$

where we made a change of the summation variables to a form more familiar in condensed matter, the traces and commutator are written using the SU(N) structure constants. Define the fermions density

$$n^a(k) := \left\langle \ \psi^a_{(k)} \check{\psi}^a_{(k)} \ \right\rangle \ . \quad (4.2)$$

We study the quasi-particle using a Hartree-Fock approximation[8]

$$E = \langle \ \mathcal{F} \ \rangle \ , \qquad \left\langle \ \psi^a_{(k)} \check{\psi}^b_{(p)} \ \right\rangle = \delta^{ab} \delta_{kp} n^a(k) \ . \quad (4.3)$$

Applying the Hartree-Fock approximation to (4.1), we find that the energy in term of the density is

$$E = g^2 \sum_{k=0}^{\infty} \Bigg[2h(k+1) \sum_a n^a(k) + \\ + \frac{1}{N} \sum_{p=0}^{\infty} \sum_{a,b,e} \frac{(p+1)\Theta(k-p)}{(k-p)(k+1)} f^e_{ab} f^e_{ba} n^b(p) n^a(k) \Bigg] \ . \quad (4.4)$$

The quasi particle energy (δE) and distribution (δn) are defined by removing the equilibrium energy and distribution

$$\delta E = E - \bar{E} \, , \qquad \delta n^a(q) := \left\langle \ \psi^a_{(q)} \check{\psi}^a_{(q)} \ \right\rangle - \bar{n}^a(q) \, , \quad (4.5)$$

where the equilibrium state for the 1-dim Fermi-sea is

$$\bar{n}^a(k) = 1 - \Theta(k - K) \ . \quad (4.6)$$

[8]See [25] for the application of a similar approximation in the microscopic study of normal Fermi liquids in condensed matter physics.

Applying the above to (4.4), we find the quasi particle energy in term of their distribution function

$$\delta E = g^2\Bigg[\sum_{k=0}^{\infty}\sum_{a}\Bigg(2h(k+1) - 2\sum_{p=0}^{K}\frac{(p+1)\Theta(k-p)}{(k-p)(k+1)}+ \\ - 2\sum_{p=0}^{K}\frac{(k+1)\Theta(p-k)}{(p-k)(p+1)}\Bigg)\delta n^a(k)+ \\ + \frac{1}{N}\sum_{k,p=0}^{\infty}\sum_{a,b,e}\frac{(p+1)\Theta(k-p)}{(k-p)(k+1)}f^e_{ab}f^e_{ba}\delta n^b(p)\delta n^a(k)\Bigg] \tag{4.7}$$

(where negative δn for momenta below the Fermi surface are the quasi-holes).

The quasi-particle energy should be measured relatively to the Fermi-level. We denote the momentum above the Fermi-surface as

$$k := K + e\ .$$

The single quasi-particle energy (for a particle above the Fermi-sea $e > 0$) is

$$\varepsilon_{q.p}(e) = 2g^2\left[h(K+e+1) - \sum_{p=0}^{K}\frac{(p+1}{(K+e-p)(K+e+1)}\right] = \tag{4.8}$$

$$= \frac{g^2}{2}\left(\frac{K+2}{K+e+1} + h(e-1)\right) \approx \frac{g^2}{2}\left[1 + h(e-1) - \frac{e-1}{K}O(K^{-2})\right]\ . \tag{4.9}$$

For completeness we also calculate the single quasi-hole energy[9] (i.e a particle below the Fermi-sea) :

$$\varepsilon_{q.h}(e) = -2g^2\Bigg[h(K-e+1) - \sum_{p=0}^{K-e-1}\frac{(p+1)}{(K-e-p)(K-e+1)} + \\ - \sum_{p=K-e+1}^{K}\frac{(K-e+1)}{(p-K+e)(p+1)}\Bigg] \approx \frac{g^2}{2}\left[1+h(e)-\frac{2+e}{K}+O(K^{-2})\right]. \tag{4.10}$$

Note that contributions to these energies come from both the quadratic piece and the interaction piece in (4.1). This is crucial as the leading $\log K$ contribution to the energy of the quasi-particle cancels between these two contributions terms. To see this, we note that the quadratic term in (4.1) by itself gives a contribution which is $\varepsilon_{q.p}(e) = 2g^2 h(K+e+1) \sim g^2 \log K$.

The fact that we found a much smaller energy suggests that the same cancelation mechanism, found in section 3 for the ground state energy, is also present for the excited states. One might therefore hope that the quasi-particles could eventually be understood on the gravity side as well. A caveat to the analysis that we carried out here is that we computed only the self energy piece - i.e. the diagonal element in the Hamiltonian in this state - and did not solve the entire mixing problem. The standard lore in Fermi liquids, however, is that mixing is possible but that the lifetime of the quasi-particles increases as the energy decreases. Of course, it could also be that the situation changes at higher orders of g^2.

[9] For a quasi-hole the energy is $\varepsilon_{q.h}(e) := -\varepsilon_{q.p}(-e)$ with $e > 0$.

5. A short discussion about 2-dim Fermi-sea

In [20] we were interested in semi-short states, invariant under a single supersymmetry charge (1/16-BPS). At the leading g behavior the supercharge changes a covariant derivative to fermion, this can be written as

$$\left\{\bar{Q}_{24}, \psi_{(n)}\right\} \sim \sum_{m=0}^{n-1} \left\{\psi_{(m)}, \psi_{(n-1+m)}\right\}$$

When considering only the action of the supercharge we constructed a 2-dim Fermi-sea operator by relaxing the condition $j_L = j_R - q_2$, for example one can have $j_L = 0$ which is the one that we will focus on. The construction is similar to the 1-dim case. Relaxing the condition on the left-handed angular moment (undotted indices) the relevant fermionic partons are

$$\begin{aligned}
\psi_{(n|0)} :=& \frac{1}{(n+1)!}(\mathbf{a}_1^\dagger)^{n+1}(\mathbf{b}_{\dot{1}}^\dagger)^n \mathbf{c}_4^\dagger \,|\, 0 \,\rangle \\
\psi_{(n|m)} :=& \frac{1}{(n+1-m)!(m)!}(\mathbf{a}_1^\dagger)^{n+1-m}(\mathbf{a}_2^\dagger)^m(\mathbf{b}_{\dot{1}}^\dagger)^n \mathbf{c}_4^\dagger \,|\, 0 \,\rangle = (\mathfrak{J}_L^-)^m \psi_{(n|0)} \,,
\end{aligned} \tag{5.1}$$

where $\mathfrak{J}_L^-$ is the $\mathrm{SU}(2)_L$ operator which lower the j_L^3 charge. It is important to notice that all left handed angular momentum indices are symmetrized for a single fermion.[10] The level of the fermion is n and it's angular momentum is

$$\left(j_L, j_L^3 \;;\; j_R, j_R^3\right) = \left(\frac{n+1}{2}, \frac{n+1}{2} - m \;;\; \frac{n}{2}, \frac{n}{2}\right) \,.$$

Notice that the components of the momentum vector obey

$$\left(\frac{n+1}{2}\right)^2 - \left(\frac{n+1}{2} - m\right)^2 \geq 0 \,.$$

Thus the momentum is confined to a 'forward light-cone', which is where the 'relativistic' nature of the Fermi-sea comes from. The $j_L = 0$ state can be built by multiplying all fermions with all values of j_L^3. This is the same as multiplying all fermions with as few derivative as possible - notice that now at each level there is a degeneracy due to different momentum vectors. Hence the $j_L = 0$ operator is

$$\mathcal{O}_{\mathrm{2dim}}^{(K)} := \prod_{n=0}^{K}\prod_{m=0}^{n+1} \mathrm{Jdet}\left(\psi_{(I|m)}\right) = \prod_{n=0}^{K}\prod_{m=0}^{n+1} \mathrm{Jdet}\left(\frac{1}{m!}(\mathfrak{J}_L^-)^m \psi_{(I|0)}\right) \,. \tag{5.2}$$

The energy[11] of this Fermi-sea is

$$\begin{aligned}
d_0 =& \sum_{n=0}^{K}\sum_{m=0}^{n+1}\sum_{a=1}^{\dim G}\left(\frac{3}{2} + n\right) = (N^2-1)\frac{(4K^2+23K+36)(K+1)}{12} \\
\approx& \frac{N^2K^3}{3} \,.
\end{aligned} \tag{5.3}$$

[10] An anti-symmetrization should be replaced, using the equation of motion, to a commutator of partons: $\epsilon^{\alpha\beta}\left[D_{\alpha \mathrm{i}}, D_{\alpha \mathrm{i}}\right] = 2\bar{F}_{\mathrm{i}\mathrm{i}}$ and $\epsilon^{\alpha\beta} D_{\alpha\mathrm{i}}\Psi_\beta = \left[\bar{\Psi}_{\mathrm{i}}^i, \Phi_{i4}\right]$. We prevent this by forcing the discussed symmetrization

[11] A straightforward computation gives $q_2 \sim N^2K^2/2$ and $j_R \sim N^2K^3/6$, corresponding to the bottom line in (1.1).

The leading order correction to the dilation operator can be calculated either by algebraic technics (see appendix E) or from the Harmonic action. We find that

$$\delta\mathfrak{D}_2 = -\frac{1}{2N}\sum_{\substack{k,k',q,q'\\ m,m',n,n'}} \mathcal{C}^{(k',m')(q',n')}_{(k,m)(q,n)}\,\mathrm{Tr}:\left\{\psi_{(k'|m')},\check{\psi}_{(k|m)}\right\}\left\{\psi_{(q'|n')},\check{\psi}_{(q|n)}\right\}: \tag{5.4}$$

Angular momentum conservation law guarantees that $k+q=k'+q'$ and $m+n=m'+n'$ - otherwise the coefficient vanishes.

When we apply $\delta\mathfrak{D}_2$ to the 2-dim Fermi-sea (5.2), Pauli's exclusion principle combined with the conservation laws guarantees that the Fermi-sea is an eigenstate, and the eigenvalue is

$$\delta\mathfrak{D}_2\left|\ \mathcal{O}^{(K)}_{2\mathrm{dim}}\ \right\rangle = (N^2-1)\sum_{k=0}^{K}\sum_{q=0}^{K}\sum_{m=0}^{k+1}\sum_{n=0}^{q+1}\mathcal{C}^{(q,n)(k,m)}_{(k,m)(q,n)}\left|\ \mathcal{O}^{(K)}_{2\mathrm{dim}}\ \right\rangle\ . \tag{5.5}$$

The only coefficient that comes into this sum is

$$\begin{aligned}\mathcal{C}^{(q,n)(k,m)}_{(k,m)(q,n)} =& +\delta_{k=q}\delta_{m=n}\left[h(k)+h(q)\right]+\\ &-\Theta(q-k)\Theta(n-m+1)\Theta(q-k-n+m+1)\frac{\binom{q-k}{n-m}\binom{k+1}{m}}{\binom{q+1}{n}}\frac{(q+1)}{(q-k)(k+1)}+\\ &-\Theta(k-q)\Theta(m-n+1)\theta(k-q-m+n+1)\frac{\binom{q+1}{n}\binom{k-q}{m-n}}{\binom{k+1}{m}}\frac{k+1}{(q+1)(k-q)}+\\ &+\frac{\binom{q+1}{n}\binom{k+1}{m}}{\binom{k+q+2}{m+n}}\frac{(k+q+2)}{(q+1)(k+1)}\,,\end{aligned} \tag{5.6}$$

and hence

$$\begin{aligned}\delta\mathfrak{D}_2\left|\ \mathcal{O}^{(K)}_{2\mathrm{dim}}\ \right\rangle =&\ (N^2-1)\sum_{k=0}^{K}\Bigg[\sum_{m=0}^{k+1}2h(k)+\\ &+\sum_{q=0}^{K}\sum_{m=0}^{k+1}\sum_{n=0}^{q+1}\frac{\binom{q+1}{n}\binom{k+1}{m}}{\binom{k+q+2}{m+n}}\frac{(k+q+2)}{(q+1)(k+1)}+\\ &-2\sum_{q=0}^{K-k-1}\sum_{m=0}^{k+1}\sum_{n=0}^{q+1}\frac{\binom{q+1}{n}\binom{k+1}{m}}{\binom{k+q+2}{m+n}}\frac{(k+q+2)}{(q+1)(k+1)}\Bigg]\left|\mathcal{O}^{(K)}_{2\mathrm{dim}}\right\rangle=\\ =&\ (N^2-1)\sum_{k=0}^{K}\Bigg[2(k+2)h(k)+\sum_{q=K-k}^{K}\frac{(k+q+3)(k+q+2)}{(k+1)(q+1)}+\\ &-2\sum_{q=0}^{K-k-1}\frac{(k+q+3)(k+q+2)}{(k+1)(q+1)}\Bigg]\left|\ \mathcal{O}^{(K)}_{2\mathrm{dim}}\ \right\rangle=\\ =&\ (N^2-1)\frac{(5K+12)(K+1)}{2}\ .\end{aligned} \tag{5.7}$$

Again we see a none-trivial cancelation of leading term of order $\sim N^2K^2\log K$. We also have a large K relation which is similar to the one obtained for the 1-dim Fermi-sea

$$\frac{\delta\mathfrak{D}_2}{d_0}\Big|\,\mathcal{O}^{(K)}_{2\mathrm{dim}}\Big\rangle \approx \frac{15}{2K}+O(K^{-2})\ . \tag{5.8}$$

However, $\delta\mathfrak{D}_2$ on these operators receives corrections at order g^3, which are under much less control than what we have discussed for the 1-dim Fermi-sea.[12] The minimal sector containing $\psi_{(k|m)}$ is the $\mathfrak{su}(1,2|3)$ sector, and hence we need to consider the full set of partons in this sector

$$\begin{aligned}
\bar{Z}_{i(n|m)} &:= \frac{1}{(n)!}(\mathfrak{J}^-_L)^m\times[(D_{1\dot{1}})^n\Phi_{4i}]\\
\bar{\Psi}^i_{(n|m)} &:= \frac{1}{(n+1)!}(\mathfrak{J}^-_L)^m\times\left[(D_{1\dot{1}})^n\bar{\Psi}^i_{\dot{1}}\right]\\
\bar{F}_{(n|m)} &:= \frac{1}{(n+2)!}(\mathfrak{J}^-_L)^m\times\left[(D_{1\dot{1}})^n\bar{F}_{\dot{1}\dot{1}}\right]
\end{aligned} \tag{5.9}$$

By conservation of the left handed angular momentum, the right handed angular momentum, the SU(4) charges and zero coupling dilatation the only processes that can mix operators are

$$\begin{aligned}
&D_{1\dot{1}}\cdots D_{2\dot{1}}\to F_{1\dot{1}}\\
&D_{1\dot{1}}\cdots \Psi_{2\,4}\to \Phi_{4i}\cdots\bar{\Psi}^i_{\dot{1}}\\
&D_{2\dot{1}}\cdots \Psi_{1\,4}\to \Phi_{4i}\cdots\bar{\Psi}^i_{\dot{1}}\\
&\Psi_{1\,4}\cdots \Psi_{2\,4}\to \Phi_{41}\cdots\Phi_{42}\cdots\Phi_{43}
\end{aligned}$$

where this notation encodes what pairs of 'letters' can be converted to which single, pair or triplet of 'letters' (at any points in the expression). All these processes expected[13] to appear at order g^3. The first kind of mixing will cause two fermions to loose momentum and therefore will be annihilated by the Pauli's exclusion principle. However all other mixing are expected to occur and the operator (5.2) is no longer an eigenstate of $\delta\mathfrak{D}$.

It is an interesting and complicated question to calculate the $\delta\mathfrak{D}_3$ terms and find an eigenstate which is close to the initial Fermi-sea. It is even more interesting and more complicated to do so to all orders. We hope to return to this problem in the future.

6. Discussion

We studied Fermi-sea operators in $\mathcal{N}=4$ SYM within perturbation theory up to order g^4. We showed that up to this order, the ground state of the fermionic $\mathfrak{su}(1,1)$ sector, and its small excitations, behave as a Fermi liquid. The correction to the anomalous dimension of this operator are suppressed by a large parameter K, which is analogous to the Fermi level.

[12] recall that this operator contains more types of 'letters'.

[13] See the discussion about the $\mathfrak{su}(2|3)\subset\mathfrak{su}(1,2|3)$ in [7]

Given that the first few orders of perturbation theory are parametrically small by this new parameter, $1/K$, it is natural to ask what is the fate of the Fermi liquid at strong coupling. One possibility is that the structure found up to g^4 is destroyed allowing $K(\log K)^m$ corrections at order g^{2n} $(m \leq n)$. The second possibility is that the function $F(g^2)$ controlling the leading $1/K$ correction (defined in (3.39)) is diverging at some finite/infinite value of g. The third and most alluring possibility is that $F(g^2)$ has a finite limit at infinitely strong coupling.

If the latter option is true than one should be able to trace the Fermi-sea operator to a classical state (geometry) in AdS. If the same holds true for operators related to the 1/16-BPS black holes in AdS_5, then it might also suggest that these black holes are not supersymmetric once α' or g_s corrections are taken into account, but rather that these corrections are suppressed at large S^5 angular momentum ($Q \sim K$).

A possible obstacle to the result are wrapping-like interaction. These appear in the dilation operator at order where the number of partons involved is of the size of the operator. In our case wrapping interaction are suppressed by g^{2N^2}, this order cannot be studied in perturbation theory.

With this caveat in mind, it will be interesting to explore the following future direction:

- In a recent paper [31], which appeared as the current paper was written, Zwiebel constructs an iterative algebraic method to calculate the dilatation operator in the $\mathfrak{psu}(1,1|2)$ sector. Using this method it may be possible[14] (in principle) to extend our calculation in the fermionic $\mathfrak{su}(1,1)$ to higher, and perhaps all, orders of g_{ym}.

- In condensed matter physics 1-dim Fermi-liquids are commonly described using bosonization, the bosonized picture allows for better control over the perturbation theory (in the coupling). It is possible to repeat the bosonization procedure in our case - for example, one possible procedure is to define bosonic rasing/lowering operators[15]

$$b^{a}_{(q)}{}^{\dagger} := \frac{i}{\sqrt{q}} \sum_{k=-\infty}^{\infty} \psi^a_{(k+q)} \check{\psi}^a_{(k)}\,, , \qquad b^{a}_{(q)} := -\frac{i}{\sqrt{q}} \sum_{k=-\infty}^{\infty} \psi^a_{(k-q)} \check{\psi}^a_{(k)} \quad . \tag{6.1}$$

  This procedure is powerful in condensed matter because in the bosonic picture the action is quadratic (the bare theory has only a 4 fermion interaction). This will not happen in our case since the higher orders in g will introduce higher order interaction in the scalars. However there may be some insight from a bosonic description.

- Another interesting issue is to generalize the construction to the two dimensional Fermi Surface (or small j_L) - in section 5 we already discussed some difficulties associated with it. The existence of supersymmetric AdS black holes with arbitrary j_L strongly motivates the existence of such a generalization but one probably needs more ingredients on top of the Fermi-sea discussed here [20]. Moreover, it could also

[14] Assuming the lift to finite N is unique.

[15] The continuation of the momentum to negative values is done formally, and should not effect the physics near the Fermi level.

be that the black holes are not exactly 1/16-BPS. This will make their identification at weak coupling even more complicated, and it will probably be possible only if they have features similar to the Fermi-seas that we discussed in this paper - i.e., the existence of a new small parameter which might suppress corrections uniformly from weak to strong coupling.

- Using a Fermi-sea construction could potentially open the way to building a host of approximately 1/16-BPS operators. For example, if we add the boson which completes the sector to the $\mathfrak{su}(1,1|1)$ sector, is easy to find additional ways to make $\delta\mathfrak{D}_2$ parametrically small in the presence of the Fermi-sea (although it is not clear how to control these in higher orders in $g_{\rm ym}$). More specifically we add the partons

$$\phi_{(n)} := \frac{1}{n!}(D_{1\dot{1}})^n \Phi_{14}\,, \tag{6.2}$$

the order g^2 anomalous dimension is

$$\begin{aligned}\delta\mathfrak{D}_2 =&\delta\mathfrak{D}_2^{(\text{fermions})}+\\ &+\frac{1}{N}\sum_{n,m,k,q=0}^{\infty}\left(\frac{\Theta(m-q)}{n+1}-\frac{\Theta(n-q)}{n-q}\right)\text{Tr}\,\boldsymbol{:}\left[\phi_{(k)},\check{\phi}_{(m)}\right]\left\{\psi_{(q)},\check{\psi}_{(n)}\right\}\boldsymbol{:}\,\delta_{n+m=k+q}+\\ &+\frac{1}{N}\sum_{n,m,k,q=0}^{\infty}\frac{\Theta(n-k+1)}{n+1}\text{Tr}\,\boldsymbol{:}\left[\phi_{(k)},\psi_{(q)}\right]\left[\check{\phi}_{(m)},\check{\psi}_{(n)}\right]\boldsymbol{:}\,\delta_{n+m=k+q}+\\ &-\frac{1}{N}\sum_{n,m,k,q=0}^{\infty}\frac{\Theta(n-q)}{n-q}\text{Tr}\,\boldsymbol{:}\left[\phi_{(k)},\check{\phi}_{(m)}\right]\left[\phi_{(q)},\check{\phi}_{(n)}\right]\boldsymbol{:}\,\delta_{n+m=k+q}+\\ &-\frac{2}{N}\sum_{m=0}^{\infty}h(m)\,\text{Tr}\,\boldsymbol{:}\phi_{(m)}\check{\phi}_{(m)}\boldsymbol{:}\ .\end{aligned} \tag{6.3}$$

An almost 1/16-BPS operator can be built by multiplying the Fermi-sea with a bosonic operator

$$\mathcal{O} = \mathcal{O}^{(K)}_{\text{1dim}}\,\text{Tr}\left(\phi^M_{(0)}\right). \tag{6.4}$$

The operator is an eigenstate of the dilatation operator (to all orders in perturbation theory) with anomalous dimension at order g^2:

$$\delta\mathfrak{D}_2\,|\,\mathcal{O}\,\rangle = \left[(N^2-1)2(K+1) - 2Mh(K+1)\right]|\,\mathcal{O}\,\rangle\ . \tag{6.5}$$

By solving $\delta\mathfrak{D}_2 = 0$ we find that for a large scalar contribution

$$M = (N^2-1)\frac{(K+1)}{h(K+1)} \sim N^2K/\log K\ . \tag{6.6}$$

Thus we can get as close to a BPS operator (since M must be an integer we cannot have an BPS state).

In the case that the large AdS black holes of [19] are indeed exactly supersymmetric (with α' and g_s corrections included) we expect a large amount (of order N^2) of

1/16-BPS semi-short multiplets. The above example demonstrates how a Fermi-sea facilitates a cancelation of anomalous dimension (due to fermion-boson interaction). A detailed search for 1/16-BPS operator following these idea and techniques is a promising but challenging avenue.

- Finally, it will be interesting to explore these techniques for application to condensed matter systems. Fermi-sea constructions are of major importance in many strongly coupled condensed matter system, but so far they were not part of any AdS/CFT model. If from $N = 4$ SYM we can learn about the description of Fermi-seas in AdS_5, we may be able to explore phenomena related to Fermi-sea in lower dimension AdS_d spaces ($d = 4, 3, 2$) as well. These in turn may be important as models for many interesting condensed matter system (superconductors, quantum hall effect etc.)

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A. The $\mathfrak{psu}(2, 2|4)$ algebra and the partons of $\mathcal{N} = 4$ SYM

In this appendix we review some properties of the $\mathfrak{psu}(2, 2|4)$ algebra and it's representations. More details can be found in [7]. The generators of the $\mathfrak{psu}(2, 2|4)$ algebra are

- The compact bosonic $\mathfrak{su}(2)_L \times \mathfrak{su}(2)_R \times \mathfrak{su}(4)$ generators $\mathfrak{L}^\alpha{}_\beta$, $\bar{\mathfrak{L}}^{\dot\alpha}{}_{\dot\beta}$, $\mathfrak{R}^a{}_b$.

- The non-compact bosonic translation, dilatation and special conformal generators $\mathfrak{P}_{\alpha\dot\alpha}$, $\mathfrak{D}$, $\mathfrak{K}^{\dot\alpha\alpha}$.

- The supercharges $\mathfrak{Q}^a_\alpha$, $\bar{\mathfrak{Q}}_{\dot\alpha a}$ and super-conformal supercharges $\mathfrak{S}^\alpha_a$, $\bar{\mathfrak{S}}^{\dot\alpha a}$.

The algebra can be extended to $\mathfrak{u}(2, 2|4)$ by introducing two additional U(1) generators (commuting with the $\mathfrak{psu}(2, 2|4)$) - the central charge $\mathfrak{C}$ and the hypercharge $\mathfrak{B}$. Physical states must satisfy $\mathfrak{C} = 0$. We use the oscillator representation to write the (zero coupling) generators, and then all commutation relations can be derived from the commutation

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relation of the oscillators (2.6)

$$
\begin{aligned}
\mathfrak{L}^{\alpha}{}_{\beta} &= \mathbf{a}^{\dagger}_{\beta}\mathbf{a}^{\alpha} - \frac{1}{2}\delta^{\alpha}_{\beta}\,\mathbf{a}^{\dagger}_{\gamma}\mathbf{a}^{\gamma} & \mathfrak{B} &= \frac{1}{2}\mathbf{a}^{\dagger}_{\gamma}\mathbf{a}^{\gamma} - \frac{1}{2}\mathbf{b}^{\dagger}_{\dot\gamma}\mathbf{b}^{\dot\gamma} \\
\bar{\mathfrak{L}}^{\dot\alpha}{}_{\dot\beta} &= \mathbf{b}^{\dagger}_{\dot\beta}\mathbf{b}^{\dot\alpha} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\,\mathbf{b}^{\dagger}_{\dot\gamma}\mathbf{b}^{\dot\gamma} & \mathfrak{D} &= 1 + \mathbf{a}^{\dagger}_{\gamma}\mathbf{a}^{\gamma} + \mathbf{b}^{\dagger}_{\dot\gamma}\mathbf{b}^{\dot\gamma} \\
\mathfrak{R}^{a}{}_{b} &= \mathbf{c}^{\dagger}_{b}\mathbf{c}^{a} - \frac{1}{2}\delta^{a}_{b}\,\mathbf{c}^{\dagger}_{c}\mathbf{c}^{c} & \mathfrak{C} &= 1 - \frac{1}{2}\mathbf{a}^{\dagger}_{\gamma}\mathbf{a}^{\gamma} + \frac{1}{2}\mathbf{b}^{\dagger}_{\dot\gamma}\mathbf{b}^{\dot\gamma} - \frac{1}{2}\mathbf{c}^{\dagger}_{c}\mathbf{c}^{c} \\
\mathfrak{Q}^{a}_{\alpha} &= \mathbf{a}^{\dagger}_{\alpha}\mathbf{c}^{a} & \bar{\mathfrak{Q}}_{\dot\alpha a} &= \mathbf{b}^{\dagger}_{\dot\alpha}\mathbf{c}^{\dagger}_{a} \\
\mathfrak{S}^{\alpha}_{a} &= \mathbf{c}^{\dagger}_{a}\mathbf{a}^{\alpha} & \bar{\mathfrak{S}}^{\dot\alpha a} &= \mathbf{b}^{\dot\alpha}\mathbf{c}^{a} \\
\mathfrak{P}_{\alpha\dot\beta} &= \mathbf{a}^{\dagger}_{\alpha}\mathbf{b}^{\dagger}_{\dot\beta} & \mathfrak{K}_{\alpha\dot\beta} &= \mathbf{a}^{\alpha}\mathbf{b}^{\dot\beta}\ .
\end{aligned} \tag{A.1}
$$

In the oscillator language representations of $\mathfrak{psu}(2,2|4)$ are labeled by a set of weights[16] $w = (d_0; j_L, j_R; q_1, p, q_2; B, L)$ which are the charges of the dilatation (zero coupling), left and right angular momenta, SU(4) weights, hypercharge, and spin-chain length. The weights are related to the Cartan generators

$$
\begin{aligned}
\mathfrak{q}_1 &= \mathfrak{R}^2{}_2 - \mathfrak{R}^1{}_1 & \mathfrak{p} &= \mathfrak{R}^2{}_2 - \mathfrak{R}^2{}_2 & \mathfrak{q}_2 &= \mathfrak{R}^4{}_4 - \mathfrak{R}^3{}_3 \\
\mathfrak{J}^3_L &= \frac{1}{2}\mathfrak{L}^1{}_1 - \frac{1}{2}\mathfrak{L}^2{}_2 & \mathfrak{J}^3_R &= \frac{1}{2}\bar{\mathfrak{L}}^{\dot 1}{}_{\dot 1} - \frac{1}{2}\bar{\mathfrak{L}}^{\dot 2}{}_{\dot 2} & & .
\end{aligned} \tag{A.2}
$$

We define two combinations of Cartan generators

$$
\begin{aligned}
\Delta_1 &= \mathfrak{D} - 2\mathfrak{J}^3_L - \frac{1}{2}\mathfrak{q}_1 - \mathfrak{p} - \frac{3}{2}\mathfrak{q}_2 \\
\Delta_2 &= \mathfrak{D} - 2\mathfrak{J}^3_R - \frac{3}{2}\mathfrak{q}_1 - \mathfrak{p} - \frac{1}{2}\mathfrak{q}_2
\end{aligned} \tag{A.3}
$$

The unitarity bounds of $\mathcal{N} = 4$ SYM are $\Delta_1 \geq 0$ and $\Delta_1 \geq 0$. For primary operators one should subtract $c_m = 0, 1, 2$ form the r.h.s. of (A.3), for 0,1 or 2 non-singlet angular momentum representations. A detailed list of the weights of all partons in presented in table 1, with all lorentz indices set to the highest-weight.

B. Gauge transformation for finite N operators

The usage of gauge transformation to generate identities between operators in the finite N language is explained in [7]. We start from the generator of gauge transformations[17]

$$
\mathfrak{j} = i \sum_C \boldsymbol{:} \big[W_C, \check{W}^C\big\} \boldsymbol{:}\ . \tag{B.1}
$$

By plugging $\mathfrak{j}$ into operational expression one can generate a large set of gauge identities. One identity we use in the main text (3.20), is proved as follows

$$
\begin{aligned}
0 \,\widehat{=}\, & - i \operatorname{Tr} \mathfrak{j} \boldsymbol{:} \big[W_A, \check{W}^B\big\} \boldsymbol{:} = \\
= & \sum_C \operatorname{Tr} \boldsymbol{:} \big[W_C, \check{W}^C\big] \big[W_A, \check{W}^B\big]\big\} \boldsymbol{:} + 2N \operatorname{Tr} \boldsymbol{:} W_A \check{W}^B \boldsymbol{:}\ ,
\end{aligned} \tag{B.2}
$$

[16] We use Fraktur font for generators and uncapitalized Latin font for weights

[17] C runs over all partons, i.e all fields with derivatives such that the lorentz indices are multiplied to form the highest possible weight.

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	d_0	$(2j_L, 2j_R)$	(q_1, p, q_2)	B	Δ_1	Δ_2
$(D_{1\dot{1}})^n \Psi_{1\,4}$	$n+3/2$	$(n+1, n)$	$(0,0,1)$	$1/2$	0	0
$(D_{1\dot{1}})^n \Phi_{34}$	$n+1$	(n, n)	$(0,1,0)$	0	0	0
$(D_{1\dot{1}})^n \Phi_{24}$	$n+1$	(n, n)	$(1,-1,1)$	0	0	0
$(D_{1\dot{1}})^n \bar{\Psi}^1_{\dot{1}}$	$n+3/2$	$(n, n+1)$	$(1,0,0)$	$-1/2$	0	0
$(D_{1\dot{1}})^n F_{11}$	$n+2$	$(n+2, n)$	$(0,0,0)$	1	0	2
$(D_{1\dot{1}})^n \Psi_{1\,3}$	$n+3/2$	$(n+1, n)$	$(0,1,-1)$	$1/2$	0	2
$(D_{1\dot{1}})^n \Psi_{1\,2}$	$n+3/2$	$(n+1, n)$	$(1,-1,0)$	$1/2$	0	2
$(D_{1\dot{1}})^n \Psi_{1\,1}$	$n+3/2$	$(n+1, n)$	$(-1,0,0)$	$1/2$	2	2
$(D_{1\dot{1}})^n \Phi_{23}$	$n+1$	(n, n)	$(1,0,-1)$	0	0	2
$(D_{1\dot{1}})^n \Phi_{14}$	$n+1$	(n, n)	$(-1,0,1)$	0	2	0
$(D_{1\dot{1}})^n \Phi_{13}$	$n+1$	(n, n)	$(-1,1,-1)$	0	2	2
$(D_{1\dot{1}})^n \Phi_{12}$	$n+1$	(n, n)	$(0,-1,0)$	0	2	2
$(D_{1\dot{1}})^n \bar{\Psi}^2_{\dot{1}}$	$n+3/2$	$(n, n+1)$	$(-1,1,0)$	$-1/2$	2	0
$(D_{1\dot{1}})^n \bar{\Psi}^3_{\dot{1}}$	$n+3/2$	$(n, n+1)$	$(0,-1,1)$	$-1/2$	2	0
$(D_{1\dot{1}})^n \bar{\Psi}^4_{\dot{1}}$	$n+3/2$	$(n, n+1)$	$(0,0,-1)$	$-1/2$	2	2
$(D_{1\dot{1}})^n \bar{F}_{\dot{1}\dot{1}}$	$n+2$	$(n, n+2)$	$(0,0,0)$	-2	2	0

Table 1: List of weights for the partons in $\mathcal{N}=4$ SYM. The symmetries related to the weights are defined in appendix A.

by restriction to the fermionic $\mathfrak{su}(1,1)$ sector we find

$$\mathrm{Tr}\,{:}\psi_{(m)}\check{\psi}_{(m)}{:} = -\frac{1}{2N}\sum_{n=0}^{\infty}\mathrm{Tr}\,{:}\left\{\psi_{(m)},\check{\psi}_{(m)}\right\}\left\{\psi_{(n)},\check{\psi}_{(n)}\right\}{:}\ . \tag{B.3}$$

The second identity we need (3.33) is proved using

$$\begin{aligned} 0 \,\widehat{=}\, & -i\,\mathrm{Tr}\,\mathfrak{j}\,{:}\left[[W_A, W_B\}, \check{W}^C\right\}{:} = \\ = & \sum_D \mathrm{Tr}\,{:}\left[W_D, \check{W}^D\right\}\left[[W_A, W_B\}, \check{W}^C\right\}{:} + 2N\,\mathrm{Tr}\,{:}W_A\left[W_B, \check{W}^C\right\}{:}\,, \end{aligned} \tag{B.4}$$

restricting to the fermionic $\mathfrak{su}(1,1)$ sector we find

$$\begin{aligned} \mathrm{Tr}\,{:}\left\{\psi_{(k)},\psi_{(q)}\right\}\check{\psi}_{(k+q+1)}{:}\ & \widehat{=} \\ & \widehat{=}\ -\frac{1}{2N}\sum_{m=0}^{\infty}\mathrm{Tr}\,{:}\left\{\psi_{(m)},\check{\psi}_{(m)}\right\}\left[\left\{\psi_{(q)},\psi_{(k)}\right\},\check{\psi}_{(k+q+1)}\right]{:}\ . \end{aligned} \tag{B.5}$$

C. Zwiebel's solution for the g^4 dilatation operator

An algebraic solution for the order g^4 dilatation operator was found by Zwiebel [29]. The solution is based on the $\mathfrak{psu}(1,1|2)$ sector. The sector is generated by the partons

$$
\begin{aligned}
\psi_{(n)} &:= \frac{1}{(n+1)!}\left(D_{1\mathrm{i}}\right)^n \Psi_{1\,4} \\
\bar{\psi}_{(n)} &:= \frac{1}{(n+1)!}\left(D_{1\mathrm{i}}\right)^n \bar{\Psi}^1_{\mathrm{i}} \\
\phi^2_{(n)} &:= \frac{1}{(n)!}\left(D_{1\mathrm{i}}\right)^n \Phi_{24} \\
\phi^3_{(n)} &:= \frac{1}{(n)!}\left(D_{1\mathrm{i}}\right)^n \Phi_{34}\ .
\end{aligned} \tag{C.1}
$$

States generated by these partons sit in representation of a $\mathfrak{psu}(1,1|2) \times \mathfrak{psu}(1|1)^2$ subalgebra of $\mathfrak{psu}(2,2|4)$

- the $\mathfrak{su}(1,1|2)$ symmetry is generated by

$$
\begin{aligned}
\mathfrak{J}^0(g) &= -\mathcal{L} + 2\mathfrak{D}_0 + \delta\mathfrak{D}(g) & \mathfrak{R}^0 &= \mathfrak{R}^2_2 - \mathfrak{R}^3_3 \\
\mathfrak{J}^{++}(g) &= \mathfrak{P}_{11}(g) & \mathfrak{J}^{--}(g) &= \mathfrak{K}^{11}(g) \\
\mathfrak{R}^{22}(g) &= \mathfrak{R}^3_2 & \mathfrak{R}^{33}(g) &= \mathfrak{R}^2_3 \\
\mathfrak{Q}^{+i}(g) &= \mathfrak{Q}^i_1(g) & \bar{\mathfrak{Q}}^{+i}(g) &= \bar{\mathfrak{Q}}_{1i}(g) \\
\mathfrak{Q}^{-i}(g) &= \bar{\mathfrak{S}}^{1i}(g) & \bar{\mathfrak{Q}}^{-i}(g) &= \mathfrak{S}^1_i(g)
\end{aligned} \tag{C.2a}
$$

  where $i = 2,3$ and $\mathfrak{R}^a_b$ are generators of the SU(4) symmetry.

- The $\mathfrak{psu}(1|1)^2$ symmetry is generated by

$$
\begin{aligned}
\mathfrak{T}^+(g) &= \bar{\mathfrak{Q}}_{24}(g) & \mathfrak{T}^-(g) &= \mathfrak{S}^2_1(g) \\
\bar{\mathfrak{T}}^+(g) &= \mathfrak{Q}^1_2(g) & \bar{\mathfrak{T}}^-(g) &= \bar{\mathfrak{S}}^{24}(g) \\
&\delta\mathfrak{D}(g) & &\mathcal{L}
\end{aligned} \tag{C.2b}
$$

With the relation

$$
\delta\mathfrak{D}(g) = 2\left\{\mathfrak{T}^+(g), \bar{\mathfrak{T}}^-(g)\right\} = 2\left\{\mathfrak{T}^-(g), \bar{\mathfrak{T}}^+(g)\right\}\ . \tag{C.3}
$$

The algebra includes the dilatation operator, thus the sector is closed under mixing. Similar to the $\mathfrak{su}(1,1)$ sector we use explicit relations between charges in the $\mathfrak{psu}(1,1|2)$ sector. A reduction of the above to the fermionic $\mathfrak{su}(1,1)$ is carried out by considering states that consists only of $\psi_{(n)}$ parton.

The first step in Zwiebel's solution is the g^1 corrections to the $\mathfrak{psu}(1|1)^2$ supercharges $\mathfrak{T}^\pm$ and $\bar{\mathfrak{T}}^\pm$, which we already discussed in the main body of the text (3.28). Next Zwiebel defines two new operators,

$$
\begin{aligned}
\mathfrak{h} := &\sum_{n=0}^{\infty} \frac{1}{2} h(n+1)\,\mathrm{Tr}\,{:}\psi_{(n)}\check{\psi}_{(n)}{:} + \sum_{n=0}^{\infty}\frac{1}{2}h(n+1)\,\mathrm{Tr}\,{:}\bar{\psi}_{(n)}\check{\bar{\psi}}_{(n)}{:} \\
&+ \sum_{n=0}^{\infty}\frac{1}{2}h(n)\sum_{i=2}^{3}\mathrm{Tr}\,{:}\phi^i_{(n)}\check{\phi}^i_{(n)}{:}
\end{aligned}
$$

$$\mathfrak{r} := \left\{\mathfrak{T}_1^-, \left[\bar{\mathfrak{T}}_1^+, \mathfrak{h}\right]\right\} - \left\{\mathfrak{T}_1^+, \left[\bar{\mathfrak{T}}_1^-, \mathfrak{h}\right]\right\} \ . \tag{C.4}$$

Using these two operators Zwiebel shows that the g^2 correction to the $\mathfrak{psu}(1,1|2)$ algebra are

$$\mathfrak{R}_2^0 = 0 \qquad \mathfrak{J}_2^0 = \delta\mathfrak{D}_2 \qquad \mathfrak{X}_2^\pm = \pm\left[\mathfrak{X}_0^\pm, \mathfrak{r}\right] + \left[\mathfrak{X}_0^\pm, \mathfrak{y}\right] , \tag{C.5}$$

with $\mathfrak{X}^\pm \in \left\{\mathfrak{J}^{++}, \mathfrak{J}^{--}, \mathfrak{Q}^{\pm i}, \bar{\mathfrak{Q}}^{\pm i}\right\}$. The g^3 corrections to the $\mathfrak{psu}(1|1)^2$ supercharges are

$$\mathfrak{T}_3^\pm = \pm\left[\mathfrak{T}_1^\pm, \mathfrak{r}\right] + \left[\mathfrak{T}_1^\pm, \mathfrak{y}\right] + \alpha\mathfrak{T}_1^\pm \qquad \bar{\mathfrak{T}}_3^\pm = \pm\left[\bar{\mathfrak{T}}_1^\pm, \mathfrak{r}\right] + \left[\bar{\mathfrak{T}}_1^\pm, \mathfrak{y}\right] + \alpha\mathfrak{T}_1^\pm \ . \tag{C.6}$$

In the above, the parameter α is related to a coupling redefinition $g \mapsto g + \alpha g^3$ and $\mathfrak{h}$ to a similarity transformation. It is possible to choose a regularization scheme such that both are zero. Finally the g^4 correction to $\delta\mathfrak{D}$ is found by anti-commutator of $\mathfrak{T}^+$ and $\bar{\mathfrak{T}}^-$ resulting in

$$\begin{aligned}\delta\mathfrak{D}_4 =& 2\left\{\mathfrak{T}^+(g), \bar{\mathfrak{T}}^-(g)\right\}_4 = \\ =& 2\left\{\mathfrak{T}_3^+, \bar{\mathfrak{T}}_1^-\right\} + 2\left\{\mathfrak{T}_1^+, \bar{\mathfrak{T}}_3^-\right\} = \\ =& 2\left\{\bar{\mathfrak{T}}_1^-, \left[\mathfrak{T}_1^+, \left\{\mathfrak{T}_1^-, \left[\bar{\mathfrak{T}}_1^+, \mathfrak{h}\right]\right\}\right]\right\} + 2\left\{\mathfrak{T}_1^+, \left[\bar{\mathfrak{T}}_1^-, \left\{\bar{\mathfrak{T}}_1^+, \left[\mathfrak{T}_1^-, \mathfrak{h}\right]\right\}\right]\right\}\end{aligned} \tag{C.7}$$

In the above we already removed the coupling redefinition and similarity transformation.[18]

D. The anomalous dimension of the 1-dim Fermi-sea

In this appendix we evaluate (3.36) on the 1-dim Fermi-sea state (3.7). First we take care of the operational structures appearing the $\delta\mathfrak{D}_4$. The key feature is that Pauli's exclusion principle combined with angular momentum conservation guarantees that all creation operators ($\psi_{(p)}$) are contracted by annihilation operators ($\check{\psi}_{(p)}$) generating delta-functions in momentum space and in the gauge group indices. The fact that the contractions are done only after the creation operator hits the Fermi-sea state results in Θ-functions limiting the summation range. The outcomes of this feature are summarized by the following identities:

$$\mathrm{Tr}\ \boldsymbol{:}\psi_{(m)}\check{\psi}_{(m)}\boldsymbol{:} \left| \ \mathcal{O}^{(K)}_{1\mathrm{dim}} \right\rangle = +\Theta(K+1-m)(N^2-1)\left| \ \mathcal{O}^{(K)}_{1\mathrm{dim}} \right\rangle \tag{D.1a}$$

$$\begin{aligned}&\mathrm{Tr}\ \boldsymbol{:}\left\{\psi_{(k)}, \check{\psi}_{(m)}\right\}\left\{\psi_{(q)}, \check{\psi}_{(n)}\right\}\boldsymbol{:} \left| \ \mathcal{O}^{(K)}_{1\mathrm{dim}} \right\rangle = \\ &\qquad = -2N(N^2-1)\delta_{q=m}\delta_{k=n}\Theta(K+1-m)\Theta(K+1-n)\left| \ \mathcal{O}^{(K)}_{1\mathrm{dim}} \right\rangle\end{aligned} \tag{D.1b}$$

$$\begin{aligned}&\mathrm{Tr}\ \boldsymbol{:}\psi_{(q)}\check{\psi}_{(n)}\psi_{(k)}\check{\psi}_{(m)}\boldsymbol{:} \left| \ \mathcal{O}^{(K)}_{1\mathrm{dim}} \right\rangle = \Theta(K+1-m)\Theta(K+1-n)\times \\ &\qquad \times \frac{(N^2-1)^2}{N}\left(\delta_{q=n}\delta_{k=m} - \delta_{q=m}\delta_{k=n}\right)\left| \ \mathcal{O}^{(K)}_{1\mathrm{dim}} \right\rangle\end{aligned} \tag{D.1c}$$

$$\begin{aligned}&\mathrm{Tr}\ \boldsymbol{:}\psi_{(q)}\check{\psi}_{(n)}\boldsymbol{:}\ \mathrm{Tr}\ \boldsymbol{:}\psi_{(k)}\check{\psi}_{(m)}\boldsymbol{:} \left| \ \mathcal{O}^{(K)}_{1\mathrm{dim}} \right\rangle = \Theta(K+1-m)\Theta(K+1-n)\times \\ &\qquad \times (N^2-1)^2\left(\delta_{q=n}\delta_{k=m} - \frac{1}{N^2-1}\delta_{q=m}\delta_{k=n}\right)\left(\left| \ \mathcal{O}^{(K)}_{1\mathrm{dim}} \right\rangle\right.\end{aligned} \tag{D.1d}$$

[18]In general there is a redefinition of generators such that $\delta\mathfrak{D}_4 \mapsto \delta\mathfrak{D}_4 + 2\alpha\delta\mathfrak{D}_2 + [\delta\mathfrak{D}_2, \mathfrak{h}]$.

$$\text{Tr} :\psi_{(q)}\psi_{(k)}\text{:} \ \text{Tr} :\check{\psi}_{m)}\check{\psi}_{(n)}\text{:} \Big| \ \mathcal{O}^{(K)}_{\text{1dim}} \ \Big\rangle = \Theta(K+1-m)\Theta(K+1-n)\times$$
$$\times \, (N^2-1)\left(\delta_{q=n}\delta_{k=m} - \delta_{q=m}\delta_{k=n}\right)\Big| \ \mathcal{O}^{(K)}_{\text{1dim}} \ \Big\rangle \quad \text{(D.1e)}$$

$$\begin{aligned}
&\text{Tr} : \left[\left\{\psi_{(r)}, \check{\psi}_{(m)}\right\}, \check{\psi}_{(l)}\right] \left[\left\{\psi_{(s)}, \check{\psi}_{(n)}\right\} \psi_{(t)}\right] : \Big| \ \mathcal{O}^{(K)}_{\text{1dim}} \ \Big\rangle = \\
&=\Theta(K+1-m)\Theta(K+1-n)\Theta(K+1-l)\, 4N^2(N^2-1)\times \\
&\times \left(-\delta_{m=s}\delta_{n=r}\delta_{l=t} + \delta_{m=s}\delta_{n=t}\delta_{l=r} + \frac{1}{2}\delta_{m=t}\delta_{n=r}\delta_{l=s}\right)\Big| \ \mathcal{O}^{(K)}_{\text{1dim}} \ \Big\rangle
\end{aligned} \quad \text{(D.1f)}$$

Combining the above result with the coefficients in (3.36) we express that g^4 anomalous dimension of the Fermi-sea by a set of sums:

$$\begin{aligned}
\delta\mathfrak{D}_4 = (N^2-1)\sum_{r=0}^{K}\sum_{s=K-r}^{K}\Bigg[&\mathcal{C}_{(n,m)}\xi_{(n,m)} - \frac{1}{2}\bar{\mu}_{(n,m)} - \frac{1}{2}\mathcal{C}_{(n,m)}\bar{\nu}_{(n,m)} + \\
&+\sum_{q=0}^{K}\left(C_{(r,s)}\nu_{[(r;r+s+1),(s;q)]} - \frac{1}{2}C_{(r,s)}\nu_{[(q;r+s+1),(s;q)]}\right) + \\
&+\sum_{q=0}^{K}\left(\mu_{[(r+s+1;r),(q;s)]} - \frac{1}{2}\mu_{[(r+s+1;q),(q;s)]}\right)\Bigg]
\end{aligned} \quad \text{(D.2)}$$

Note that we did not applied any large N limit, the non-planar contributions (the double trace part of $\delta\mathfrak{D}_4$) has the same order of magnitude in N as the planar contributions. We were unable to calculate the expression above analytically, instead we use a numerical calculation. Even thou the expression is quite complicated the series is easily found numerically

$$\delta\mathfrak{D}_4 \Big| \ \mathcal{O}^{(K)}_{\text{1dim}} \ \Big\rangle = -(N^2-1)(2K+2)\Big| \ \mathcal{O}^{(K)}_{\text{1dim}} \ \Big\rangle \ . \quad \text{(D.3)}$$

We checked this result for all values of K up to $K = 150$, and found a perfect match.

E. The g^2 anomalous dimension of the 2-dim Fermi-sea

The fermions $\psi_{(k|m)}$ transform in a representation of $\mathfrak{su}(1,2)$, up to order g^2 this is a closed sector however at order g^2 they mix within the larger $\mathfrak{su}(1,2|3)$ sector. We will focus on the g^2 correction to the dilatations operator allowing us to focus on the smaller symmetry. The algebra which closes on the fermions is $\mathfrak{su}(1,2)\times\mathfrak{u}(1|1)$.

- The $\mathfrak{su}(1,2)$ algebra

$$\begin{aligned}
\mathfrak{J}'_{++}(g) &= \mathfrak{K}^{2\dot{1}}(g) & \mathfrak{J}'_{-+}(g) &= \mathfrak{P}_{\dot{1}1}(g) \\
\mathfrak{J}'_{+-}(g) &= \mathfrak{K}^{1\dot{1}}(g) & \mathfrak{J}'_{--}(g) &= \mathfrak{P}_{\dot{1}2}(g) \\
\mathfrak{J}'_{0}(g) &= -\mathfrak{D} - \frac{1}{2}\left(\bar{\mathfrak{L}}^1_1 - \bar{\mathfrak{L}}^2_2\right) & \mathfrak{L}'_{+} &= \mathfrak{L}^2_1 \\
\mathfrak{L}'_{-} &= \mathfrak{L}^1_2 & \mathfrak{L}'_{0} &= \frac{1}{2}\left(\mathfrak{L}^1_1 - \mathfrak{L}^2_2\right)
\end{aligned} \quad \text{(E.1a)}$$

- The $\mathfrak{u}(1|1)$ algebra

$$\mathcal{L} \qquad \mathfrak{Q}'_-(g) = \bar{\mathfrak{Q}}_{\dot{2}4}(g) \qquad \mathfrak{Q}'_+(g) = \bar{\mathfrak{S}}^{\dot{2}4}(g) \qquad \delta\mathfrak{D}(g) \tag{E.1b}$$

The commutation relation (inherited from the $\mathfrak{psu}(2,2|4)$ algebra) are

$$\left[\mathcal{L}, \mathfrak{Q}'_\pm\right] = \mp\mathfrak{Q}'_\pm \quad \left\{\mathfrak{Q}'_+, \mathfrak{Q}'_-\right\} = \frac{1}{2}\delta\mathfrak{D} \quad [\mathcal{L}, \delta\mathfrak{D}] = 0 \quad \left[\delta\mathfrak{D}, \mathfrak{Q}'_\pm\right] = 0 \tag{E.1c}$$

The $\mathfrak{su}(1,2)$ and $\mathfrak{u}(1|1)$ generators commute with each other. At the leading order in g, the generator of $\mathfrak{su}(1,2)$ are the free theory (g^0) generators. For the $\mathfrak{u}(1|1)$ the supercharges are of order g^1 and $\delta\mathfrak{D}$ is of order g^2. The operation of $\mathfrak{su}(1,2)$ on the spin states is

$$\begin{aligned}
\mathfrak{J}'_{++} \mid (k,m) \rangle &= k\,\delta_{m\neq 0} \mid (k-1, m-1) \rangle \\
\mathfrak{J}'_{--} \mid (k,m) \rangle &= (m+1) \mid (k+1, m+1) \rangle \\
\mathfrak{L}'_{+} \mid (k,m) \rangle &= (k-m+2)\,\delta_{m\neq 0} \mid (k, m-1) \rangle \\
\mathfrak{L}'_{-} \mid (k,m) \rangle &= (m+1)\,\delta_{k-m+1\neq 0} \mid (k, m+1) \rangle \ .
\end{aligned} \tag{E.2}$$

For the supercharge we make the ansatz (most general form that conserves the charges)

$$\begin{aligned}
\mathfrak{Q}'_- \mid (k,m) \rangle &= \sum_{q=0}^{k-1} \sum_{n=\max(0,m+q-k)}^{\min(q+1,m)} c^-_{q,n;k,m} \mid (q,n)(k-q-1, m-n) \rangle \\
\mathfrak{Q}'_+ \mid (q,n)(k-q, m-n) \rangle &= c^+_{q,n;k,m} \mid (k+1, m) \rangle \ .
\end{aligned} \tag{E.3}$$

Next we prove that the algebra forces that

$$c^-_{q,n;k,m} = a_- \ ,$$

up to some unknown constant a_-. The key ingredient is that the $\mathfrak{u}(1|1)$ and $\mathfrak{su}(1,2)$ factors must commute also in perturbation theory (non zero coupling) however the commutation is up to a term that vanishes due to the gauge structure. We need to check that the ansatz (E.3) is consistent with the algebra, i.e the supercharges commutes with the generators of $\mathfrak{su}(1,2)$ up to gauge transformations.[19] Both supercharges conserves $\mathfrak{J}'_0$ and $\mathfrak{L}'_0$ trivially. For the other generators we find

$$\begin{aligned}
&\left[\mathfrak{Q}'_-, \mathfrak{L}'_+\right] \mid (k,m) \rangle = \\
&\quad = \sum_{q=0}^{k-1} \sum_{n=\max(0,m+q-k-1)}^{\min(q+1,m-1)} \Big[\delta_{m\neq 0}(k-m+2)c^-_{q,n;k,m-1} + \\
&\qquad\qquad - c^-_{q,n+1;k,m}(q-n+1) + \\
&\qquad\qquad - c^-_{q,n;k,m}(k-q-m+n+1)\Big] \\
&\qquad \times \mid (q,n)(k-q-1, m-n-1) \rangle
\end{aligned} \tag{E.4a}$$

[19] More correctly, any operator with the correct charges that vanishes when the trace is applied to the spin chain can appear in the commutation of the supercharges and $\mathfrak{su}(1,2)$.

$$
\begin{aligned}
\big[\mathfrak{Q}'_-,\mathfrak{L}'_-\big]\,|\,(k,m)\,\rangle &= \\
&= \sum_{q=0}^{k-1}\ \sum_{\substack{n=\\ \max(0,m+q-k+1)}}^{\min(q+1,m+1)} \Big[c^-_{q,n;k,m+1}\delta_{k-m+1\neq 0}(m+1) - c^-_{q,n-1;k,m}n+ \\
&\qquad\qquad - c^-_{q,n;k,m}(m-n+1)\Big] \\
&\quad \times |\,(q,n)(k-q-1,m-n+1)\,\rangle
\end{aligned} \tag{E.4b}
$$

$$
\begin{aligned}
\big[\mathfrak{Q}'_-,\mathfrak{J}'_{-+}\big]\,|\,(k,m)\,\rangle &= \\
&= \sum_{q=0}^{k}\ \sum_{\substack{n=\\ \max(0,m+q-k-1)}}^{\min(q+1,m)} \Big[(k+2-m)c^-_{q,n;k+1,m} - c^-_{q-1,n;k,m}\delta_{q\neq 0}(q-n+1)+ \\
&\qquad\qquad - c^-_{q,n;k,m}\delta_{q\neq k}(k-q-m+n+1)\Big] \\
&\quad \times |\,(q,n)(k-q,m-n)\,\rangle
\end{aligned} \tag{E.4c}
$$

$$
\begin{aligned}
\big[\mathfrak{Q}'_-,\mathfrak{J}'_{+-}\big]\,|\,(k,m)\,\rangle &= \\
&= \sum_{q=0}^{k-2}\ \sum_{\substack{n=\\ \max(0,m+q-k+1)}}^{\min(q+1,m)} \Big[\delta_{k-m+1\neq 0}kc^-_{q,n;k-1,m} - c^-_{q+1,n;k,m}(q+1)+ \\
&\qquad\qquad - c^-_{q,n;k,m}(k-q-1))\Big] \\
&\quad \times |\,(q,n)(k-q-2,m-n)\,\rangle
\end{aligned} \tag{E.4d}
$$

$$
\begin{aligned}
\big[\mathfrak{Q}'_-,\mathfrak{J}'_{--}\big]\,|\,(k,m)\,\rangle &= \\
&= \sum_{q=0}^{k}\ \sum_{\substack{n=\\ \max(0,m+q-k)}}^{\min(q+1,m+1)} \Big[(m+1)c^-_{q,n;k+1,m+1} - c^-_{q-1,n-1;k,m}\delta_{q\neq 0}n+ \\
&\qquad\qquad - c^-_{q,n;k,m}\delta_{q\neq k}(m-n+1)\Big] \\
&\quad \times |\,(q,n)(k-q,m-n+1)\,\rangle
\end{aligned} \tag{E.4e}
$$

$$
\begin{aligned}
\big[\mathfrak{Q}'_-,\mathfrak{J}'_{++}\big]\,|\,(k,m)\,\rangle &= \\
&= \sum_{q=0}^{k-2}\ \sum_{\substack{n=\\ \max(0,m+q-k)}}^{\min(q+1,m-1)} \Big[\delta_{m\neq 0}kc^-_{q,n;k-1,m-1} - c^-_{q+1,n+1;k,m}(q+1)+ \\
&\qquad\qquad - c^-_{q,n;k,m}(k-q-1)\Big] \\
&\quad \times |\,(q,n)(k-q-2,m-n-1)\,\rangle\ .
\end{aligned} \tag{E.4f}
$$

Demanding the the r.h.s. vanishes we find that $c^-_{q,n;k,m} = a_-$. Indeed the commutators vanish up to gauge transformations

$$
\big[\mathfrak{Q}'_-,\mathfrak{L}'_+\big]\,|\,(k,m)\,\rangle = 0 \tag{E.5a}
$$

$$
\big[\mathfrak{Q}'_-,\mathfrak{L}'_-\big]\,|\,(k,m)\,\rangle = 0 \tag{E.5b}
$$

$$\big[\mathfrak{Q}'_-, \mathfrak{J}'_{-+}\big] \mid (k,m)\ \rangle = a_-\Big(\mid (0,0)(k,m)\ \rangle + \mid (k,m)(0,0)\ \rangle\Big) \tag{E.5c}$$

$$\big[\mathfrak{Q}'_-, \mathfrak{J}'_{+-}\big] \mid (k,m)\ \rangle = 0 \tag{E.5d}$$

$$\big[\mathfrak{Q}'_-, \mathfrak{J}'_{--}\big] \mid (k,m)\ \rangle = \delta_{k\neq 0} a_-\Big(\mid (0,1)(k,m)\ \rangle + \mid (k,m)(0,1)\ \rangle\Big) \tag{E.5e}$$

$$\big[\mathfrak{Q}'_-, \mathfrak{J}'_{++}\big] \mid (k,m)\ \rangle = 0\ . \tag{E.5f}$$

Note: One may start with the most general gauge transformation on the right hand side of the commutators. Using the consistency of the algebra and the physical demand that a single fermion (with no derivative) cannot split due to the supercharge.[20]

The second supercharge has the commutation relations

$$\begin{aligned}\big[\mathfrak{Q}'_+, \mathfrak{L}'_+\big] \mid (q,n)(k-q,m-n)\ \rangle = \\ = \Big[&\delta_{n\neq 0}(q-n+2)c^+_{q,n-1;k,m-1}+ \\ &+\delta_{m-n\neq 0}(k-q-m+n+2)c^+_{q,n;k,m-1}+ \\ &-\delta_{m\neq 0}(k-m+3)c^+_{q,n;k,m}\Big] \mid (k+1,m-1)\ \rangle\end{aligned} \tag{E.6a}$$

$$\begin{aligned}\big[\mathfrak{Q}'_+, \mathfrak{L}'_-\big] \mid (q,n)(k-q,m-n)\ \rangle = \\ = \Big[&\delta_{q-n+1\neq 0}(n+1)c^+_{q,n+1;k,m+1}+ \\ &+\delta_{k-q-m+n+1\neq 0}(m-n+1)c^+_{q,n;k,m+1}+ \\ &-\delta_{k-m+2\neq 0}(m+1)c^+_{q,n;k,m}\Big] \mid (k+1,m+1)\ \rangle\end{aligned} \tag{E.6b}$$

$$\begin{aligned}\big[\mathfrak{Q}'_+, \mathfrak{J}'_{-+}\big] \mid (q,n)(k-q,m-n)\ \rangle = \\ = \Big[&(q-n+2)c^+_{q+1,n;k+1,m} \\ &+(k-q-m+n+2)c^+_{q,n;k+1,m}+ \\ &-(k-m+3)c^+_{q,n;k,m}\Big] \mid (k+2,m)\ \rangle\end{aligned} \tag{E.6c}$$

$$\begin{aligned}\big[\mathfrak{Q}'_+, \mathfrak{J}'_{+-}\big] \mid (q,n)(k-q,m-n)\ \rangle = \\ = \Big[&\delta_{q-n+1\neq 0}\, q c^+_{q-1,n;k-1,m} \\ &+\delta_{k-q-m+n+1\neq 0}(k-q)c^+_{q,n;k-1,m}+ \\ &-\delta_{k-m+2\neq 0}(k+1)c^+_{q,n;k,m}\Big] \mid (k,m)\ \rangle\end{aligned} \tag{E.6d}$$

$$\begin{aligned}\big[\mathfrak{Q}'_+, \mathfrak{J}'_{--}\big] \mid (q,n)(k-q,m-n)\ \rangle = \\ = \Big[&(n+1)c^+_{q+1,n+1;k+1,m+1} \\ &+(m-n+1)c^+_{q,n;k+1,m+1}+ \\ &-(m+1)c^+_{q,n;k,m}\Big] \mid (k+2,m+1)\ \rangle\end{aligned} \tag{E.6e}$$

$$\big[\mathfrak{Q}'_+, \mathfrak{J}'_{++}\big] \mid (q,n)(k-q,m-n)\ \rangle =$$

[20] Unlike $\psi_{k|m}$, a single Ψ_{14} without derivative is not a composite operator and there is no renormalization that can cause a change of the supercharge from the tree-level supercharge.

$$
\begin{aligned}
=\Big[&\delta_{n\neq 0}\, q\, c^{+}_{q-1,n-1;k-1,m-1} \\
&+\delta_{m-n\neq 0}(k-q)c^{+}_{q,n;k-1,m-1}+ \\
&-\delta_{m\neq 0}(k+1)c^{+}_{q,n;k,m}\Big] \mid (k,m-1)\ \rangle\ .
\end{aligned}
\tag{E.6f}
$$

All the commutators above must vanish up to a gauge transformation. The unique solution is

$$
c^{+}_{q,n;k,m} = \frac{(k+2)}{(q+1)(k-q+1)} \frac{\binom{q+1}{n}\binom{k-q+1}{m-n}}{\binom{k+2}{m}}\, a_{+}\,, \tag{E.7}
$$

at which the commutators take the value

$$
\begin{aligned}
\big[\mathfrak{Q}'_{+}, \mathfrak{L}'_{+}\big] \mid (q,n)(k-q,m-n)\ \rangle &= 0 && \text{(E.8a)}\\
\big[\mathfrak{Q}'_{+}, \mathfrak{L}'_{-}\big] \mid (q,n)(k-q,m-n)\ \rangle &= 0 && \text{(E.8b)}\\
\big[\mathfrak{Q}'_{+}, \mathfrak{J}'_{-+}\big] \mid (q,n)(k-q,m-n)\ \rangle &= 0 && \text{(E.8c)}\\
\big[\mathfrak{Q}'_{+}, \mathfrak{J}'_{+-}\big] \mid (q,n)(k-q,m-n)\ \rangle &= -a_{+}\left(\delta_{q=0}+\delta_{q=k}\right) \mid (k,m+1)\ \rangle && \text{(E.8d)}\\
\big[\mathfrak{Q}'_{+}, \mathfrak{J}'_{--}\big] \mid (q,n)(k-q,m-n)\ \rangle &= 0 && \text{(E.8e)}\\
\big[\mathfrak{Q}'_{+}, \mathfrak{J}'_{++}\big] \mid (q,n)(k-q,m-n)\ \rangle &= 0\ . && \text{(E.8f)}
\end{aligned}
$$

Summarizing the above we find

$$
\mathfrak{Q}'_{+} \mid (q,n)(k,m)\ \rangle = a_{+} \frac{(k+q+2)}{(q+1)(k+1)} \frac{\binom{q+1}{n}\binom{k+1}{m}}{\binom{k+q+2}{m+n}} \mid (k+q+1, m+n)\ \rangle \tag{E.9}
$$

$$
\mathfrak{Q}'_{-} \mid (k,m)\ \rangle = a_{-} \sum_{q=0}^{k-1} \sum_{\substack{n=\\ \max(0,m+q-k)}}^{\min(q+1,m)} \mid (q,n)(k-q-1,m-n)\ \rangle\ . \tag{E.10}
$$

The 1-loop spin-chain Hamiltonian ($\delta\mathfrak{D}_2$) can be calculated from the commutator

$$
\delta\mathfrak{D} = 2\big\{\mathfrak{Q}'_{+}, \mathfrak{Q}'_{-}\big\}\ . \tag{E.11}
$$

Acting on a two-spin state. First we calculate

$$
\begin{aligned}
\mathfrak{Q}'_{-} \mid (q,n)(k,m)\ \rangle = &+ a_{-} \sum_{s=0}^{q-1} \sum_{\substack{l=\\ \max(0,n+s-q)}}^{\min(s+1,n)} \mid (s,l)(q-s-1,n-l)(k,m)\ \rangle + \\
&- a_{-} \sum_{s=0}^{k-1} \sum_{\substack{l=\\ \max(0,m+s-k)}}^{\min(s+1,m)} \mid (q,n)(s,l)(k-s-1,m-l)\ \rangle\ .
\end{aligned}
\tag{E.12}
$$

In the spin chain limit $\mathfrak{Q}'_{+}$ acts only on adjacent partons, thus we find

$$
\mathfrak{Q}'_{+}\mathfrak{Q}'_{-} \mid (q,n)(k,m)\ \rangle = a_{+}a_{-}\Bigg\{
$$

$$+\frac{1}{2}\sum_{s=0}^{q-1}\sum_{\substack{l=\\ \max(0,n+s-q)}}^{\min(s+1,n)}\frac{(q+1)}{(s+1)(q-s)}\frac{\binom{s+1}{l}\binom{q-s}{n-l}}{\binom{q+1}{n}}\mid (q,n)(k,m)\ \rangle+$$

$$-\sum_{s=0}^{q-1}\sum_{\substack{l=\\ \max(0,n+s-q)}}^{\min(s+1,n)}\frac{(k+q-s+1)}{(k+1)(q-s)}\frac{\binom{q-s}{n-l}\binom{k+1}{m}}{\binom{k+q-s+1}{m+n-l}}\mid (s,l)(k+q-s,m+n-l)\ \rangle+$$

$$-\sum_{s=q+1}^{k+q}\sum_{\substack{l=\\ \max(n,n+m+s-1-k-q)}}^{\min(n+s-q,m+n)}\frac{(s+1)}{(q+1)(s-q)}\frac{\binom{q+1}{n}\binom{s-q}{l-n}}{\binom{s+1}{l}}\mid (s,l)(k+q-s,m+n-l)\ \rangle+$$

$$+\frac{1}{2}\sum_{s=0}^{k-1}\sum_{\substack{l=\\ \max(0,m+s-k)}}^{\min(s+1,m)}\frac{(k+1)}{(s+1)(k-s)}\frac{\binom{s+1}{l}\binom{k-s}{m-l}}{\binom{k+1}{m}}\mid (q,n)(k,m)\ \rangle\Bigg\}\ . \tag{E.13}$$

The factor of $\frac{1}{2}$ comes due to symmetrization of the operator as explained in [7] for the $\mathfrak{su}(1,1)$ case.

Acting with the commutator (E.11) on a single spin state we find

$$\mathfrak{Q}'_-\mathfrak{Q}'_+\mid (q,n)(k,m)\ \rangle\ = a_-a_+\sum_{s=0}^{k+q}\sum_{\substack{l=\\ \max(0,m+n+s-k-q-1)}}^{\min(s+1,m+n)}\times$$

$$\times\frac{(k+q+2)}{(q+1)(k+1)}\frac{\binom{q+1}{n}\binom{k+1}{m}}{\binom{k+q+2}{m+n}}\mid (s,l)(k+q-s,m+n-l)\ \rangle. \tag{E.14}$$

Applying (E.13), (E.14) to (E.11) we find (setting $2a_+a_-=1$)

$$\begin{aligned}\delta\mathfrak{D}_2\mid (q,n)(k,m)\ \rangle =&\Big[h(q)+h(k)\Big]\mid (q,n)(k,m)\ \rangle+\\ &+\Bigg[-\sum_{s=0}^{q-1}\sum_{\substack{l=\\ \max(0,n+s-q)}}^{\min(s+1,n)}\frac{(k+q-s+1)}{(k+1)(q-s)}\frac{\binom{q-s}{n-l}\binom{k+1}{m}}{\binom{k+q-s+1}{m+n-l}}+\\ &-\sum_{s=q+1}^{k+q}\sum_{\substack{l=\\ \max(n,n+m+s-1-k-q)}}^{\min(n+s-q,m+n)}\frac{(s+1)}{(q+1)(s-q)}\frac{\binom{q+1}{n}\binom{s-q}{l-n}}{\binom{s+1}{l}}+\\ &+\sum_{s=0}^{k+q}\sum_{\substack{l=\\ \max(0,m+n+s-k-q-1)}}^{\min(s+1,m+n)}\frac{(k+q+2)}{(q+1)(k+1)}\frac{\binom{q+1}{n}\binom{k+1}{m}}{\binom{k+q+2}{m+n}}\Bigg]\\ &\times\mid (s,l)(k+q-s,m+n-l)\ \rangle\end{aligned} \tag{E.15}$$

the lifting of the spin chain Hamiltonian to finite N is done using (2.12) and (2.14)

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